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The RQE-CAPM : New insights about the pricing of idiosyncratic risk

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ABSTRACT

We use an equivalent form of Markowitz's mean-variance utility function, based on Rao's Quadratic Entropy (RQE), to enrich the standard capital asset pricing model (CAPM), both in the presence and in the absence of a risk-free asset. The resulting equilibrium, which we denote RQE-CAPM, offers important new insights about the pricing of risk. Notably, it reveals that the reason for which the standard CAPM does not price idiosyncratic risk is not only because the market portfolio is law of large numbers diversified but also because the model implicitly assumes agents' total risk aversion and their correlation diversification risk preference balance each other exactly. We then demonstrate that idiosyncratic risk is priced in a general RQE-CAPM where agents' total risk aversion and their correlation diversification risk preference coefficients are not necessary equal. Our general RQE-CAPM therefore offers a unifying way of thinking about the pricing of idiosyncratic risk, including cases where such risk is negatively priced, and is relevant for the literature assessing the idiosyncratic risk puzzle. It also provides a natural theoretical underpinning for the empirical tests of the CAPM or the pricing of idiosyncratic risk performed in some existence studies.

Keywords: Rao's Quadratic Entropy, Mean-Variance Model, Capital Asset Pricing Model, Idiosyncratic Risk, Correlation Diversification.

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1 Introduction

The capital asset pricing model (CAPM), inspired from the seminal work of Markowitz (1952) and independently developed by Sharpe (1964), Lintner (1965), Mossin (1966) and Treynor (1961), marks the birth of asset pricing theory. It is considered "...one of the two or three major contributions of academic research to financial managers during the post-war era" (Jagannathan and Wang, 1996). Its main prediction is that only market risk is priced while idiosyncratic risk is not because it is *law of large numbers* diversified. Despite several criticisms, both theoretical and empirical, the CAPM remains finance's main tool to assess the cost of capital, measure portfolio performance and diversification and evaluate investment strategies.[1,2]

This paper provides a new disaggregation analysis of the CAPM, both in the presence (Sharpe's version) and in the absence (Black's (1972) version) of a risk-free asset, that sheds new light on the pricing of idiosyncratic risk and provides a general theoretical explanation of the idiosyncratic risk puzzle mentioned in the literature. Central to our derivations is the representation of Markowitz's (1952) mean-variance utility function based on Rao's Quadratic Entropy (RQE), a general statistical tool developed by Rao (1982a,b) to measure population diversity and used in fields such as statistics (see Nayak, 1986a,b; Rao, 1982a,b), ecology (see Champely and Chessel, 2002; Pavoine, 2012; Pavoine and Bonsall, 2009; Pavoine et al., 2005; Ricotta and Szeidl, 2006; Zhao and Naik, 2012), energy policy (see Stirling, 2010), economics (see Nayak and Gastwirth, 1989), innovation intellectual property (see Khachatryan and Muehlmann, 2019) and portfolio theory (see Carmichael et al., 2015, 2018).

First documented in Carmichael et al. (2015), this equivalent representation of Markowitz's (1952) mean-variance utility function is obtained by disaggregating portfolio variance into two types of risk: total risk — as measured by the weighted average of asset variances — and *correlation* diversification risk — as measured by RQE. This disaggregation highlights how,

[1] A portfolio is *law of large numbers* diversified if it is both diversified in terms of size and weights; see Koumou (2020) for more details.

[2] For theoretical criticisms of the CAPM, we refer readers to Markowitz (2005) and Goltz and Le Sourd (2011). For empirical criticisms, see Friend and Blume (1970), Miller and Scholes (1972), Black et al. (1972), Blume and Friend (1973), Kraus and Litzenberger (1976), Roll (1977), Levy (1978), Tinic and West (1984), Tinic and West (1986), Lehmann (1990), Fama and French (1992), Fama and French (1993), Malkiel and Xu (1997), Goyal and Santa-Clara (2003), Malkiel and Yexiao (2006), Ang et al. (2006), Mitton and Vorkink (2007), Ang et al. (2009), Boyer et al. (2010) and Fu (2009), among others.

with Markowitz's (1952) mean-variance utility function, risk-averse agents are sensitive to the two types risk and how they dislike total risk but like correlation diversification risk, a distinction hidden in the standard mean-variance utility function. Looking at this divergent attitude of agents towards these two types of risk, one might naturally ask whether they have any bearing on the pricing of risks, notably idiosyncratic risk. *A priori*, one may guess that total risk would be priced positively (because it is disliked by agents) and correlation diversification risk negatively (because it is liked by agents). However, it is difficult to guess the implications of agents' attitude towards total risk and correlation diversification risk in terms of the pricing of idiosyncratic risk (because it is both a part of total risk and correlation diversification risk, therefore both liked and disliked by agents); more theoretical investigation is required. Hence our reexamination of asset pricing in the light of the RQE representation of Markowitz's (1952) mean-variance utility function.[3]

Our paper makes three contributions. First, we update the standard CAPM, both in the presence (Sharpe's version) and in the absence (Black's version) of a risk-free asset, using the RQE representation of Markowitz's (1952) mean-variance utility function. Our new derivation, which we denote RQE-CAPM, is a two-beta linear CAPM. The first beta is related to an asset's total risk (total risk beta) and is a scaled measure obtained by dividing the asset's variance by the total risk of the market portfolio. The second beta is related to an asset's correlation diversification risk (correlation diversification risk beta) and is obtained by dividing the asset's correlation diversification risk in the market portfolio by its market counterpart; the aggregation of these two betas naturally yields the standard CAPM's beta. Geometrically, the RQE-CAPM reveals that the standard CAPM security market line is located in a plane indexed by the total risk and correlation diversification risk betas. We identify this plane as the *RQE security market plane*.

Our second contribution is to provide new insights about the pricing mechanism at play in the standard CAPM. We show that the total risk beta is positively priced while its correlation diversification risk counterpart is negatively priced, and that the aggregation of these two opposite premiums yields the standard CAPM premium. The RQE-CAPM thus highlights how agents are first compensated for assets' total risk and are then willing to

[3]A *correlation* diversification is a diversification principle that exploits the interdependence between assets to reduce risk; see Koumou (2020) for more details. The term *correlation* refers here to any measure of dependency.

pay, in terms of a reduced return, for assets' correlation diversification risk. The market total risk premium can therefore be interpreted as a gross premium, the market correlation diversification risk premium as a tax and the standard CAPM's market risk premium as a net premium. This interpretation can be extended to the betas.

Relatedly, the RQE-CAPM shows that idiosyncratic risk is priced twice (positively because it is a risk and negatively because it is a correlation diversification risk) but that these two effects cancel each other in the standard CAPM because agents' total risk aversion and correlation diversification risk preference coefficients are equal in Markowitz's (1952) mean-variance utility function. As such, the reason for which idiosyncratic risk is not priced in the standard CAPM is not only because the market portfolio is *law of large numbers* diversified, but also because agents' total risk aversion and correlation diversification risk preference coefficients are implicitly assumed to be equal. A natural question that arises is whether idiosyncratic risk is priced when this implicit assumption is relaxed.

Our third contribution answers this question. To do so, we rederive the RQE-CAPM, both in the presence and in the absence of risk-free asset, when agents' total risk aversion and correlation diversification risk preference coefficients are not necessary equal. This general RQE-CAPM predicts that idiosyncratic risk *is* priced provided that the societal ratio of total risk aversion to correlation diversification risk preference coefficients is not equal to 1. More specifically, when the ratio is strictly greater than 1, idiosyncratic risk is positively priced whereas it is priced negatively when the ratio is strictly lower than 1, with the standard result of no price on idiosyncratic risk obtaining when the ratio is 1.

The RQE-CAPM therefore offers a unifying way of thinking about the pricing of idiosyncratic risk. As such, it is relevant for the idiosyncratic risk puzzle documented in studies such as Ang et al. (2006), Ang et al. (2009), Guo and Savickas (2010) and Boyer et al. (2010). Many contributions [4] have put forth possible rationalizations of the puzzle, with partially success (Hou and Loh, 2016). The RQE-CAPM is an important preference-based potential explanation of this puzzle, in which investors' affinity for high idiosyncratic volatility stems from a preference for correlation diversification and a desire to hold a more correlation di-

[4] See Johnson (2004), Barberis and Huang (2008), Bali and Cakici (2008), Boyer et al. (2010), Huang et al. (2009), Fu (2009), Jiang et al. (2009), Bali et al. (2011), Han and Lesmond (2011), Chen and Petkova (2012), Han and Kumar (2013), Bhootra and Hur (2015), Stambaugh et al. (2015), Cao and Han (2016), Shi et al. (2016), Gu et al. (2018), Egginton and Hur (2018) and Aslanidis et al. (2019) among others.

versified portfolio, rather than an inclination towards speculation or gambling, as in other contributions (Bali et al., 2011; Barberis and Huang, 2008; Boyer et al., 2010; Han and Kumar, 2013; Mitton and Vorkink, 2007).

We also show that our general RQE-CAPM offers testable cross-sectional implications, some of which have already been assessed by empirical studies, including Levy (1978), Malkiel and Yexiao (2006), Lakonishok and Shapiro (1986), Friend and Westerfield (1981), Lakonishok and Shapiro (1984), Carroll and Wei (1988), Cadsby (1992), Lehmann (1990) and Amihud and Mendelson (1989). As such, our general RQE-CAPM delivers theoretical underpinnings to these empirical explanations on the pricing of idiosyncratic risk.

This paper is organized as follows. Section 2 reviews the standard CAPM (Section 2.1) as well as RQE and its application to portfolio selection (Section 2.2). Section 3 presents our update of the standard CAPM based on the RQE representation of the Markowitz's (1952) mean-variance utility function. Section 4 develops our general RQE-CAPM. Section 5 concludes.

2 Preliminaries

This section provides a brief review of the capital asset pricing model and Rao's Quadratic Entropy.

2.1 Capital Asset Pricing Model

The standard formulation of the capital asset pricing model (CAPM) assumes an economy populated by K atomistic price takers trading N risky assets in Black's (1972) version or N risky assets plus one risk-free asset in Treynor (1961), Sharpe (1964), Lintner (1965) and Mossin's (1966) versions. It is a single-period model in which each agent k allocates optimally her/his wealth W_k among available assets using a mean-variance motive and then receives *expost* stochastic investment payoffs. For convenience, let individual wealth W_k be expressed in proportion of aggregate invested wealth such that $\sum_{k=1}^{K} W_k = 1$. The investment problem for agent k is therefore

$$\max_{\mathbf{w}_k \in \mathbb{R}^N} \mathbf{w}_k^\top \boldsymbol{\mu} - \frac{\mathbf{w}_k^\top \boldsymbol{\Sigma}\, \mathbf{w}_k}{\tau_k} \text{ s.t } \sum_{i=1}^{N} w_{ki} = 1, \tag{1}$$

where $\mathbb{R}$ is the set of real numbers, $\mathbf{w}_k = (w_{k1}, \cdots, w_{k\overline{N}})^\top$ is the vector the fractions of agent k's wealth w_{ki} allocated to assets, $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_{\overline{N}})^\top$ is the vector of assets' expected returns, $\boldsymbol{\Sigma} = (\sigma_{ij})_{i=1}^{\overline{N}}$ is the covariance matrix of asset returns, $\frac{1}{\tau_k}$ is agent k's risk aversion coefficient (with τ_k her/his risk tolerance coefficient) and $\top$ is the transpose operator. The subscript $\overline{N}$ is equal to N when the risk-free asset is absent and to $N+1$ when it is present, with $N+1$ identifying the risk-free asset.

All information is costlessly shared among agents. There are no transaction costs nor capital or income taxes, and all assets are perfectly divisible and liquid. Finally, there are no short sales restrictions and when a risk-free asset is available its rate is an exogenous variable. Readers are referred to Jensen (1972), Harris (1980) and Levy (2012) for further details about the CAPM assumptions.

In Sharpe (1964), Lintner (1965) and Mossin's (1966) formulations, agent's k efficient portfolio is a weighted combination of the risk-free asset and the risky portfolio $\mathbf{w}_m = (w_{m1}, \cdots, w_{mN})^\top$ having typical element $w_{mi} = \sum_{k=1}^{K} W_k \, w_{ki}$. It can be shown that $\mathbf{w}_m$ is the tangent market portfolio in equilibrium; this is the well-known separation theorem first demonstrated in Tobin (1958). More importantly, the equilibrium expected return on asset i is

Sharpe's CAPM

$$\mu_i = \mu_{N+1} + (\mu_m - \mu_{N+1})\beta_i, \;\; \forall\, i = 1, \cdots, N, \tag{2}$$

where $\mu_m = \sum_{i=1}^{N} w_{mi}\, \mu_i$ is the expected market return and $\beta_i = \sigma_{mi}/\sigma_m^2$ the market beta of asset i, measured as the covariance of asset i's return with its market counterpart divided by the variance of the market return, which is commonly interpreted as either the marginal contribution of asset i to the market portfolio risk, or the covariance risk of asset i in the market portfolio.

Equation (2) is referred to as the Security Market Line (SML). It states that, in equilibrium, the expected return on any asset is equal to the risk-free rate plus a risk premium, with the latter being equal to the market premium, $\lambda \equiv \mu_m - \mu_{N+1}$, times asset's market beta, β_i.

From (2), one can derive the relationship between the realized return on asset i, r_i, and that of the market, r_m,

$$r_i = \mu_{N+1} + \beta_i \left(r_m - \mu_{N+1}\right) + e_i, \;\; \forall\, i = 1, \cdots, N, \tag{3}$$

where e_i is uncorrelated with r_m and has a zero expected value. It follows that

$$\sigma_i^2 = \beta_i^2\, \sigma_m^2 + \sigma_{e_i}^2, \;\; \forall\, i = 1, \cdots, N, \tag{4}$$

where the first term, $\beta_i^2\, \sigma_m^2$, is identified as asset i's systematic (or undiversifiable) risk and represents the portion of the total risk of investing in asset i associated with the market as a whole, while the second term, $\sigma_{e_i}^2$, is asset i's idiosyncratic (or unsystematic) risk. When N is large and w_{mi} is evenly distributed (in other words when the market portfolio $\mathbf{w}_m$ is *law of large numbers* diversified), only the systematic risk is remunerated by the market. This occurs because unsystematic risk should be eliminated through this diversification strategy.[5]

In the absence of a risk-free asset, Black (1972) demonstrates that risk averse agents hold an efficient portfolio formed as a weighted combination of the market portfolio and the minimum-variance zero-β portfolio. The latter is defined as the minimum variance portfolio among all portfolios uncorrelated with the market portfolio. The equilibrium expected return on asset i becomes

Black's CAPM

$$\mu_i = \mu_z + (\mu_m - \mu_z)\beta_i, \;\; \forall\, i = 1, ..., N, \tag{5}$$

where μ_z is the expected return on any zero-β portfolio. Equation (5) is analogous to (2) except for the zero-β portfolio return replacing the risk-free rate.

2.2 Rao's Quadratic Entropy

We briefly review both the general definition of Rao's Quadratic Entropy, its portfolio selection version and its relation with Markowitz's (1952) mean-variance utility function. We follow Rao (1982a,b) for the general definition and Carmichael et al. (2015, 2018) for

[5]The assumption "N is large and w_{mi} is evenly distributed" together imply that the term $\sigma_i^2\, w_{mi}$ in $\beta_i = \frac{\sigma_i^2\, w_{mi} + \sum_{j \neq i}^{N-1} \sigma_{ij} w_{mj}}{\Sigma_m}$ can be neglected which is not the case when N is small or w_{mi} is not evenly distributed.

the application to the context of portfolio selection.

2.2.1 General Definition

Given a population $\mathcal{P}$ of M individuals, RQE is defined as the average difference between two randomly drawn individuals from $\mathcal{P}$. Formally, suppose that each individual in $\mathcal{P}$ is characterized by a set of measurement X and denote by P the probability distribution function of X. Further suppose that X is a discrete random variable. Then RQE of $\mathcal{P}$ is defined as:

$$H(\mathbf{p}) = \mathbf{p}^\top \mathbf{D}\, \mathbf{p}, \tag{6}$$

where $\mathbf{p}$ is a column vector of probabilities with elements $p_i = P(X = x_i)$, $\forall\, i = 1, \cdots, M$ and $\mathbf{D} = (d_{ij})_{i,j=1}^{M}$ is a non-negative symmetric dissimilarity matrix with typical element d_{ij} expressing the difference between individuals i and j. The interpretation of RQE is straightforward: the higher is $H(\mathbf{p})$, the higher is the diversity of individuals among $\mathcal{P}$.

2.2.2 Portfolio Definition

The definition of RQE in the context of portfolio theory, introduced by Carmichael et al. (2015), is obtained by transposing the magnitudes $\mathcal{P}$, X, P and $\mathbf{D}$ to a portfolio selection setting. Consider the universe of assets as a population $\mathcal{P}$ of $\overline{N}$ assets. Next, define the random variable X to take the finite values $1, \cdots, \overline{N}$ ($\overline{N}$ assets) and its probability distribution to coincide with the weight distribution of the assets (i.e. $P(X = i) = w_i$, $\forall\, i = 1, ..., \overline{N}$), so that it is associated to the random experiment wherein assets are randomly selected (with replacement) from portfolio $\mathbf{w} = \left(w_1, \cdots, w_{\overline{N}}\right)^\top$. Then RQE is defined as half of the mean difference between two randomly drawn (with replacement) assets from portfolio $\mathbf{w}$:

$$H(\mathbf{w}) = \frac{1}{2}\, \mathbf{w}^\top \mathbf{D}\, \mathbf{w}, \tag{7}$$

where $\mathbf{D} = (d_{ij})_{i,j=1}^{\overline{N}}$ is a dissimilarity matrix between the various assets of the portfolio.

Carmichael et al. (2015) show that when $\mathbf{D}$ is suitable chosen, RQE becomes a valid class of portfolio diversification measures, able to efficiently summarize complex features of portfolio diversification and that provides a unifying measure of correlation diversification that embeds many previous contributions. Its interpretation is straightforward. All things equal, the higher $H(\mathbf{w})$ is, the more correlation diversified portfolio $\mathbf{w}$ is, because the more dissimilar assets are, the less is the probability that they do poorly at the same time and in the same proportion.[6]

2.2.3 Markowitz's (1952) Mean-Variance Utility Function

Carmichael et al. (2015) also reframe Markowitz's (1952) mean-variance utility function in terms of the RQE. Consider the following decomposition of portfolio variance $\sigma^2(\mathbf{w})$

$$\sigma^2(\mathbf{w}) = \mathbf{w}^\top \boldsymbol{\sigma}^2 - \left(\mathbf{w}^\top \boldsymbol{\sigma}^2 - \sigma^2(\mathbf{w})\right) \tag{8}$$

with $\boldsymbol{\sigma}^2 = \left(\sigma_i^2, \cdots, \sigma_N^2\right)^\top$ the vector containing each assets' variance. The first term, $\mathbf{w}^\top \boldsymbol{\sigma}^2$, is portfolio total risk. The second, $\mathbf{w}^\top \boldsymbol{\sigma}^2 - \sigma^2(\mathbf{w})$, is a well-established measure of portfolio diversification known as *diversification returns* (Bouchey et al., 2012; Chambers and Zdanowicz, 2014; Qian, 2012; Willenbrock, 2011), or *excess growth rate* (Fernholz, 2010). In addition, Carmichael et al. (2015) demonstrate that it is the specific portfolio diversification measure in Markowitz's (1952) mean-variance model, and it is equal to a specific RQE; formally

$$\mathbf{w}^\top \boldsymbol{\sigma}^2 - \sigma^2(\mathbf{w}) = H(\mathbf{w}), \tag{9}$$

where the dissimilarity matrix associated to $H(\mathbf{w})$ is $\mathbf{D} = (d_{ij})_{i,j=1}^{N}$ such that $d_{ij} = \sigma_i^2 + \sigma_j^2 - 2\,\sigma_{ij}$.[7]

Replacing $\mathbf{w}^\top \boldsymbol{\sigma}^2 - \sigma^2(\mathbf{w})$ from (9) into (8), we obtain

$$\sigma^2(\mathbf{w}) = \mathbf{w}^\top \boldsymbol{\sigma}^2 - H(\mathbf{w}). \tag{13}$$

[6] In the presence of short sales, the weight of asset i, w_i, can not longer be interpreted as probability. However, $H(\mathbf{w})$ remains well-defined and its interpretation as a class of correlation diversification measures remains valid.

[7] The dissimilarity $d_{ij} = \sigma_i^2 + \sigma_j^2 - 2\,\sigma_{ij}$ can be viewed as a generalization of Wright's (1987) relatively negative expectation dependent concept which is a necessary and sufficient condition of diversification in the mean-variance model. To see this, Wright (1987, Theorem 3.2, pp. 115) shows that when (R_i, R_j) is normally distributed R_i is (strictly) RNED on R_j, which it is denoted (strict) RNED$(R_i|R_j)$, if

$$\mathrm{Cov}(R_i - R_j, R_j) \leq (<) 0 \tag{10}$$

for every increasing function for which the covariance is defined. RNED$(R_i|R_j)$ is not symmetric, where Cov(., .) is the covariance operator. One can define the symmetric version of RNED$(R_i|R_j)$ as follows: R_i is (strictly) relatively negative expectation dependent on R_j, which it is denoted (strict) RNED(R_i, R_j), if

$$\mathrm{Cov}(R_i - R_j, R_j) + \mathrm{Cov}(R_j - R_i, R_i) \leq (<) 0 \tag{11}$$

for every increasing function for which the covariance is defined. It is clearly observable that

$$\mathrm{RNED}(R_i, R_j) \Longrightarrow d_{ij} \geq (>) 0 \tag{12}$$

This implies that d_{ij} can be considered as a generalization of the concept of RNED. The second term $(\mathbf{w}^\top \boldsymbol{\sigma}^2 - \sigma^2(\mathbf{w}))$ captures therefore the correlation diversification.

The decomposition (13) states that portfolio variance is equal to the difference between portfolio total risk and portfolio correlation diversification risk (i.e. a risk which can be diversified through correlation diversification). Thus, portfolio variance can be interpreted as portfolio risk which can not be diversified through correlation diversification.

Replacing $\sigma^2(\mathbf{w})$ from (13) in Markowitz's (1952) mean-variance utility function, we now obtain its RQE representation

$$U(\mathbf{w}) = \mathbf{w}^\top \boldsymbol{\mu} - \frac{1}{\tau}\mathbf{w}^\top \boldsymbol{\sigma}^2 + \frac{1}{\tau}H(\mathbf{w}). \tag{14}$$

The utility function (14) is an equivalent and disaggregated form of Markowitz's (1952) mean-variance utility function. It emphasizes two types of risk, total risk and correlation diversification risk, which have different significance for risk-averse agents. They dislike total risk, but like correlation diversification risk. Moreover, they have same sensitivity to both risks as $\frac{1}{\tau}$ represents the agents' total risk aversion coefficient as well as their correlation diversification risk preference coefficient. As a result, in Markowitz's mean-variance model, not only risk aversion is equivalent to preference for diversification (Dekel, 1989), but they also balance each other exactly. In this paper, we study the asset pricing implications of the utility (14) both when the two coefficients are equal and when they are different.

3 RQE-CAPM

In this section, we use the utility function (14) to derive the equilibrium in capital markets both in the presence and in the absence of a risk-free asset. We then discuss the implications of our new equilibrium in terms of asset risk pricing.

3.1 Equilibrium

The RQE representation of Markowitz's (1952) mean-variance utility function (14) modifies agent k's investment problem (1) as follows:

$$\max_{\mathbf{w}_k \in \mathbb{R}^{\overline{N}}} \mathbf{w}_k^\top \left(\boldsymbol{\mu} - \frac{\boldsymbol{\sigma}^2}{\tau_k}\right) + \frac{\mathbf{w}_k^\top \mathbf{D}\, \mathbf{w}_k}{2\tau_k} + \nu_k \left(1 - \sum_{i=1}^{\overline{N}} w_{ki}\right), \tag{15}$$

where ν_k is the Lagrange multiplier of the investment constraint, which can be interpreted as agent k's marginal utility of scaled wealth.

Using this modified investment problem for each agent k, we now derive the market equilibrium following Sharpe (1991) and starting with the case $\overline{N} = N$ where the risk-free asset is absent. The first order conditions of problem (15) are

$$\mu_i - \frac{\sigma_i^2}{\tau_k} + \frac{D_{ki}}{\tau_k} = \nu_k, \ \ \forall\, i = 1, ..., N, \tag{16}$$

where $D_{ki} = \sum_{j=1}^{N} w_{kj} d_{ij}$ measures the dissimilarity of asset i with agent k's optimal portfolio. Assuming market-clearing, the equilibrium relationships among key variables are obtained by aggregating individual optimality conditions and taking into account the relative amounts of wealth, W_k, each has invested. To do so, first rewrite (16) as

$$\tau_k\, \mu_i - \sigma_i^2 + D_{ki} = \nu_k \tau_k, \ \ \forall\, i = 1, ..., N. \tag{17}$$

Next, multiply (17) by W_k and sum the resulting equations over all agents to obtain the following set of N conditions, which must hold in equilibrium:

$$\sum_{k=1}^{K} W_k\, \tau_k\, \mu_i - \sum_{k=1}^{K} W_k\, \sigma_i^2 + \sum_{k=1}^{K} W_k\, D_{ki} = \sum_{k=1}^{K} W_k\, \nu_k\, \tau_k, \ \ \forall\, i = 1, ..., N. \tag{18}$$

By denoting $\tau_m = \sum_{k=1}^{K} W_k\, \tau_k$ the societal risk tolerance coefficient (which happens to equate the societal correlation diversification risk aversion coefficient) and $\nu_m = \frac{\sum_{k=1}^{K} W_k\, \nu_k \tau_k}{\tau_m}$ the societal marginal utility of wealth, and recalling that $\sum_{k=1}^{K} W_k = 1$, (18) can be rewritten concisely as

$$\mu_i - \frac{\sigma_i^2}{\tau_m} + \frac{D_{mi}}{\tau_m} = \nu_m, \ \ \forall\, i = 1, ..., N, \tag{19}$$

where $D_{mi} = \sum_{j=1}^{N} d_{ij} w_{mj}$ with $w_{mi} = \sum_{k=1}^{K} W_k\, w_{ki}$ is the dissimilarity of asset i with the market. The term D_{mi} denotes the diversification gain obtained at the margin by putting an additional unit of asset i in the market portfolio $\mathbf{w}_m$. Note that this gain can be decomposed in two components:

$$D_{mi} = \sum_{j=1}^{N} \left(\sigma_i^2 - \sigma_{ij}\right) w_{mj} + \sum_{j=1}^{N} \left(\sigma_j^2 - \sigma_{ij}\right) w_{mj}, \ \ \forall\, i = 1, ..., N, \tag{20}$$

where the first component, $\sum_{j=1}^{N} \left(\sigma_i^2 - \sigma_{ij}\right) w_{mj}$, is the gain in terms of the risk reduction of asset i, and the second component, $\sum_{j=1}^{N} \left(\sigma_j^2 - \sigma_{ij}\right) w_{mj}$, is the risk reduction stemming from all other assets.

Next, use (19) to isolate the expected return on asset i:

$$\mu_i = \nu_m + \frac{\sigma_i^2}{\tau_m} - \frac{D_{mi}}{\tau_m}, \;\; \forall\, i = 1, ..., N, \tag{21}$$

This shows that in equilibrium there is a linear relationship between the expected returns on assets, their variances and their dissimilarities with the market portfolio. Following the literature, it is convenient to express this relationship in terms of two betas, the first, β_{vi}, associated to total risk and the second, β_{hi}, to correlation diversification risk, as follows:

RQE-CAPM in the absence of a risk-free asset

$$\mu_i = \nu_m + \lambda_v \beta_{vi} + \lambda_h \beta_{hi}, \;\; \forall\, i = 1, ..., N, \tag{22}$$

where

$$\lambda_v = \frac{\bar{\sigma}_m^2}{\tau_m}, \tag{23}$$

$$\beta_{vi} = \frac{\sigma_i^2}{\bar{\sigma}_m^2}, \tag{24}$$

$$\lambda_h = -\frac{2H_m}{\tau_m}, \tag{25}$$

$$\beta_{hi} = \frac{D_{mi}}{2H_m} \tag{26}$$

with $\bar{\sigma}_m^2 = \sum_{i=1}^N w_{mi}\sigma_i^2$ the market's total risk and $H_m = H(\mathbf{w}_m)$ its RQE.

Equation (22) is a disaggregated form of the Black's CAPM equilibrium derived using the RQE representation of Markowitz's (1952) mean-variance utility function. To facilitate its comparison with the standard Black's CAPM, we express it in terms of the expected return on a zero-beta portfolio, μ_z. Multiplying (22) by w_{zi} and summing over i, we obtain

$$\mu_z = \nu_m + \lambda_v\, \beta_{vz} + \lambda_h\, \beta_{hz}, \tag{27}$$

where $\beta_{vz} = \sum_{i=1}^N w_{zi}\beta_{vi}$ and $\beta_{hz} = \sum_{i=1}^N w_{zi}\beta_{hi}$. Subtracting (27) from (22) and rearranging the resulting equation gives our alternative formulation of (22)

$$\mu_i = \mu_z + \lambda_v(\beta_{vi} - \beta_{vz}) + \lambda_h(\beta_{hi} - \beta_{hz}), \;\; \forall\, i = 1, ..., N. \tag{28}$$

When a risk-free asset is present, μ_z must equal μ_{N+1}, β_{vz} must equal $\beta_{v(N+1)} = 0$ and β_{hz} must equal $\beta_{h(N+1)}$. Under these conditions, our new representation of the Sharpe's CAPM equilibrium derived using the RQE representation of the Markowitz's (1952) mean-variance utility function is

RQE-CAPM in the present of a risk-free asset

$$\mu_i = \mu_{N+1} + \lambda_v\beta_{vi} + \lambda_h(\beta_{hi} - \beta_{h(N+1)}), \;\; \forall\, i = 1, ..., N. \tag{29}$$

As expected, our new representation of the standard CAPM, which we denote RQE-CAPM, is a two-beta linear CAPM. First β_{vi} is the total risk beta, a scaled measure obtained by dividing an asset's variance by the market portfolio's total risk. Second β_{hi} is the correlation diversification risk beta, a scaled measure obtained by dividing the correlation diversification risk of an asset in the market portfolio by its market's counterpart.

Combining (28) and (29) with their counterparts from the standard CAPM (5) and (2), the following relation between the beta, β_i, in the standard CAPM and the betas, β_{vi} and β_{hi}, in the RQE-CAPM can be established:

Relation between betas in the absence of a risk-free asset

$$\beta_i = \frac{\lambda_v}{\lambda}(\beta_{vi} - \beta_{vz}) + \frac{\lambda_h}{\lambda}(\beta_{hi} - \beta_{hz}), \;\; \forall\, i = 1, ..., N \tag{30}$$

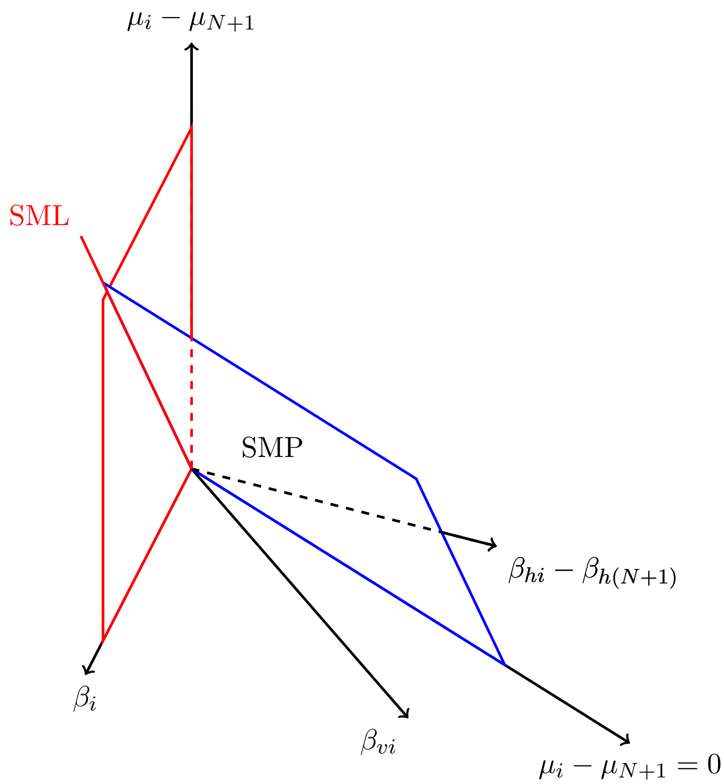
Relation between betas in the presence of a risk-free asset

$$\beta_i = \frac{\lambda_v}{\lambda}\beta_{vi} + \frac{\lambda_h}{\lambda}(\beta_{hi} - \beta_{h(N+1)}), \;\; \forall\, i = 1, ..., N. \tag{31}$$

Relations (30) and (31) show, as expected, that β_i, is the premium-weighted average of β_{vi} and β_{hi}.

The RQE-CAPM expressed by (28) and (29) can be also be depicted graphically as a three-dimensional graph showing how in equilibrium, an asset's expected return is related to both risk dimensions. For example, in the case where a risk-free asset is present, as in (29), Figure 1 provides an illustration. Assets are located along the security market line, which is itself located in a plane, named RQE Security Market Plane (RQE-SMP). Note that the two risk dimensions are not orthogonal.

Figure 1: RQE Security Market Plane (RQE-SMP) in the presence of a risk-free asset: $\lambda_v = 1$, $\lambda_h = -1$



3.2 Pricing of Risk

We now discuss the risk pricing implications of the RQE-CAPM. As wealth $W_k > 0$, $\forall\, k$, the societal risk tolerance τ_m is also positive, which implies that λ_v, the market price (or premium) for an asset's total risk β_{vi}, is positive. In addition, it implies that λ_h, the market price (or premium) for an asset's correlation diversification risk β_{hi}, is negative. As a consequence, the RQE-CAPM predicts that an asset's total risk β_{vi} is priced positively while its correlation diversification risk β_{hi} is priced negatively; in other words, the RQE-CAPM highlights that agents are first compensated for assets' total risk and then are willing to pay, in terms of a reduced return, for assets' correlation diversification risk.

Parallel interpretations for the risk premia are as follows: the market total risk premium

λ_v can be interpreted as a gross premium, while its market correlation diversification counterpart, λ_h, is a market tax, and the standard CAPM market risk premium, λ, is a net premium balancing λ_v and λ_h. To see this multiply (28) by w_{mi} and sum over i to obtain

$$\mu_m - \mu_z = \lambda = (\lambda_v + \lambda_h) - (\lambda_v\, \beta_{vz} + \lambda_h\, \beta_{hz}). \tag{32}$$

Substituting $\beta_{vz} = \frac{\sigma_z^2}{\bar{\sigma}_m^2}$, $\beta_{hz} = \frac{\sigma_z^2 + \sigma_m^2}{2H_m}$ with σ_m^2 the variance of the market portfolio, λ_v from (23) and λ_h from (25) into the second term, $\lambda_v\, \beta_{vz} + \lambda_h\, \beta_{hz}$, and rearranging, we obtain the following relation between the premiums λ, λ_v and λ_h

Relation between the premiums in the absence of a risk-free asset

$$\lambda = 2\, \lambda_v + \lambda_h. \tag{33}$$

The relation (33) shows that the standard CAPM market premium, λ, is in fact the sum of two times the market total risk premium, λ_v, and the market correlation diversification risk premium λ_h highlighted by our RQE-CAPM formulation. This relation remains valid when a risk-free asset is available.

We now examine the specific case of the pricing of idiosyncratic risk. Since idiosyncratic risk is diversification risk, the RQE-CAPM implies that it is priced twice: positively because it is a risk and negatively (taxed) because it is a correlation diversification risk. Importantly, (33) also shows that these two effects cancel each other in the standard case because agents' total risk aversion and correlation diversification risk preference coefficients are implicitly equal (recall (14)). To see this (and without loss of generality) consider the case where a risk-free asset is present. First substituting σ_i^2 from (4) into β_{vi} from (24), one gets

$$\beta_{vi} = \frac{\beta_i^2 \sigma_m^2}{\bar{\sigma}_m^2} + \frac{\sigma_{e_i}^2}{\bar{\sigma}_m^2}. \tag{34}$$

Next, from (26), one can verify that

$$\beta_{hi} - \beta_{h(N+1)} = \frac{\sigma_i^2}{2H_m} - \frac{2\sigma_{mi}}{2H_m} \tag{35}$$

with $\sigma_{mi} = \sum_{j=1}^{N} \sigma_{ij} w_{mj}$. Substituting σ_i^2 from (4) into $\beta_{hi} - \beta_{h(N+1)}$ from (35), one gets

$$\beta_{hi} - \beta_{h(N+1)} = \frac{\beta_i^2 \sigma_m^2}{2H_m} - \frac{2\sigma_{mi}}{2H_m} + \frac{\sigma_{e_i}^2}{2H_m}. \tag{36}$$

Denote by $\beta_{vi}^I = \frac{\sigma_{e_i}^2}{\bar{\sigma}_m^2}$ and $\beta_{hi}^I = \frac{\sigma_{e_i}^2}{2H_m}$ the idiosyncratic components, and by $\beta_{vi}^S = \frac{\beta_i^2 \sigma_m^2}{\bar{\sigma}_m^2}$ and $\beta_{hi}^S = \frac{\beta_i^2 \sigma_m^2}{2H_m} - \frac{2\sigma_{mi}}{2H_m} = (\beta_i^2 - 2\beta_i) \frac{\sigma_m^2}{2H_m}$ the systematic components in (35) and (36). The premium λ_v^I on the idiosyncratic risk component β_{vi}^I is

$$\lambda_v^I = \lambda_v \, \beta_{vi}^I = \frac{\sigma_{e_i}^2}{\tau_m},$$

while the premium λ_h^I on the idiosyncratic risk component β_{hi}^I is

$$\lambda_h^I = \lambda_h \, \beta_{hi}^I = -\frac{\sigma_{e_i}^2}{\tau_m}.$$

The two premiums clearly sum to zero. As a result, in the standard CAPM, idiosyncratic risk is not priced, not only because the market portfolio is *law of large numbers* diversified, but also, perhaps more importantly, because the model implicitly assumed that agents' total risk aversion and correlation diversification risk preference coefficients are equal. This finding raises the following question: should idiosyncratic risk be priced when agents' total risk aversion coefficient and their correlation diversification risk preference coefficient are not equal? In the next section, we answer this question by developing a general RQE-CAPM, both in the absence and in the presence of a risk-free asset, in which these two coefficients are not necessary equal.

4 General RQE-CAPM

Our general RQE-CAPM is based on an economy similar in all aspects to one analyzed above except for the utility function, which is now assumed to be, without loss of generality and to facilitate the comparison with the RQE-CAPM, as follows:

$$U(\mathbf{w}_k) = \mathbf{w}_k^\top \boldsymbol{\mu} - \frac{1}{\varsigma_k} \mathbf{w}_k^\top \boldsymbol{\sigma}^2 + \frac{1}{\tau_k} H(\mathbf{w}_k), \tag{37}$$

so that, relative to (15), the total risk aversion coefficient, $\frac{1}{\varsigma_k}$, is no longer necessary equal to the correlation diversification risk preference coefficient, $\frac{1}{\tau_k}$. Define $\iota_k = \frac{\tau_k}{\varsigma_k}$ as agent k's risk aversion scaled by its appetite for diversification. When $\iota_k < 1$ ($\iota_k > 1$), her/his preference for correlation diversification risk, measured by $H(\mathbf{w}_k)$, is higher (lower) than her/his aversion for total risk, measured by $\mathbf{w}_k^\top \boldsymbol{\sigma}^2$. In other words, agent k's appetite to reduce risk through correlation diversification is higher (lower) than that to reduce risk through concentration on low-variance assets. Her/his optimal portfolio will therefore be biased towards assets with high potential correlation diversification (low variance). The utility function (37) can be helpful for thinking about diversification puzzles (see Bianchi, 2018; Campbell, 2006; Goetzmann and Kumar, 2008; Lozza et al., 2018; Mitton and Vorkink, 2007; Polkovnichenko, 2005) as well as puzzles about idiosyncratic risk (see Ang et al., 2006, 2009; Boyer et al., 2010; Guo and Savickas, 2010). In what follows, we demonstrate its ability to explain idiosyncratic risk puzzle.[8]

4.1 Equilibrium

Under the general utility function (37), agent k's investment problem becomes the following, counterpart to (15) in the case $\overline{N} = N$ where the risk-free asset is absent:

$$\max_{\mathbf{w}_k \in \mathbb{R}^{\overline{N}}} \mathbf{w}_k^\top \left(\boldsymbol{\mu} - \frac{\boldsymbol{\sigma}^2}{\varsigma_k} \right) + \frac{\mathbf{w}_k^\top \mathbf{D}\, \mathbf{w}_k}{2\tau_k} + \nu_k^* \left(1 - \sum_{i=1}^{N} w_{ki} \right), \tag{38}$$

where ν_k^* is the Lagrange multiplier of the investment constraint. The first order conditions of (38) are

$$\mu_i - \frac{\sigma_i^2}{\varsigma_k} + \frac{D_{ki}}{\tau_k} = \nu_k^*, \ \ \forall\, i = 1, ..., N. \tag{39}$$

Equation (39) can be rewritten as follows

$$\tau_k \mu_i - \frac{\tau_k}{\varsigma_k} \sigma_i^2 + D_{ki} = \tau_k \nu_k^*, \ \ \forall\, i = 1, ..., N. \tag{40}$$

Multiplying (40) by W_k and summing the resulting equation over all agents, as was done in Section 3.1 above (see (18)), allows to obtain a set of N conditions that must hold in equilibrium:

$$\mu_i - \frac{\iota_m}{\tau_m} \sigma_i^2 + \frac{D_{mi}}{\tau_m} = \nu_m^*, \ \ \forall\, i = 1, ..., N, \tag{41}$$

[8] Asset i has a high potential correlation diversification if its total dissimilarity $\sum_{j=1}^N d_{ij} = \sum_{j=1}^N \left(\sigma_i^2 + \sigma_j^2 - 2\sigma_{ij} \right)$ is high. Thus, assets with high potential correlation diversification are those whose returns have high variance and low correlations. Inversely, assets with low potential correlation diversification are those with low variance and high correlations.

where, as before, $\frac{1}{\tau_m}$ is the societal correlation diversification risk preference coefficient with $\tau_m = \sum_{k=1}^{K} W_k \, \tau_k$ and $\nu_m^* = \frac{\sum_{k=1}^{K} W_k \, \tau_k \nu_k^*}{\tau_m}$ is the societal marginal utility of wealth, but now with $\iota_m = \sum_{k=1}^{K} W_k \, \iota_k = \sum_{k=1}^{K} W_k \, \frac{\tau_k}{\varsigma_k}$ the societal ratio of total risk aversion coefficient to correlation diversification risk preference coefficient. This ratio will be key in our discussion below on the pricing of idiosyncratic risk.

From (41) and following the steps used above to obtain (28), the equilibrium expected return on asset i in our general RQE-CAPM is, in the absence of a risk-free asset:

General RQE-CAPM in the absence of a risk-free asset

$$\mu_i = \mu_z + \lambda_v^* \, (\beta_{vi} - \beta_{vz}) + \lambda_h \, (\beta_{hi} - \beta_{hz}), \;\; \forall \, i = 1, ..., N. \tag{42}$$

with

$$\lambda_v^* = \iota_m \, \frac{\bar{\sigma}_m^2}{\tau_m} = \iota_m \, \lambda_v. \tag{43}$$

When a risk-free asset is present by contrast, from (41) and following the steps used above to obtain (29), the equilibrium expected return on asset i becomes:

General RQE-CAPM in the presence of a risk-free asset

$$\mu_i = \mu_{N+1} + \lambda_v^* \, \beta_{vi} + \lambda_h \, (\beta_{hi} - \beta_{h(N+1)}), \;\; \forall \, i = 1, ..., N. \tag{44}$$

The interpretation of our general RQE-CAPM is similar to that of the RQE-CAPM, except that the total risk premium is now $\lambda_v^* = \iota_m \, \lambda_v$, which differs from one obtained above in (28) and (29) only by the factor ι_m. As such, our general RQE-CAPM implies that the standard CAPM (or its RQE-CAPM representation) misprice assets' total risk, and the mispricing coefficient is the societal ratio of total risk aversion coefficient to correlation diversification risk preference coefficient ι_m. Specifically, the standard CAPM underprices (overprices) assets' total risk if $\iota_m < 1$ ($\iota_m > 1$). [9]

Geometrically, our general RQE-CAPM states that all assets are located along an amended

[9] It is important to note that (39) can also be rewritten as follows

$$\varsigma_k \mu_i - \sigma_i^2 + \frac{\varsigma_k}{\tau_k} D_{ki} = \varsigma_k \nu_k^*, \;\; \forall \, i = 1, ..., N. \tag{45}$$

In that case the mispricing will concern both total and correlation diversification risks, but unfortunately it is not explicitly identifiable.

security market line, which is itself located in a amended security market plane. The amended security market lines are

Amended security line in the absence of a risk-free asset

$$\mu_i = \mu_z + \lambda\, \beta_i^*,$$

Amended security line in the presence of a risk-free asset

$$\mu_i = \mu_{N+1} + \lambda\, \beta_i^*$$

with

$$\beta_i^* = \frac{\lambda_v^*}{\lambda}\beta_{vi} + \frac{\lambda_h}{\lambda}\left(\beta_{hi} - \beta_{h(N+1)}\right).$$

Figure 2 (Figure 3) offers an illustration of our general RQE-CAPM equilibrium in the presence of a risk-free asset in the case of $\iota_m > 1$ ($\iota_m < 1$).

Figure 2: Security Market Plane (SMP): $\lambda_v^* = 1.5$, $\lambda_h = -1$, $\iota_m = 1.5$, $\lambda = 1.5$

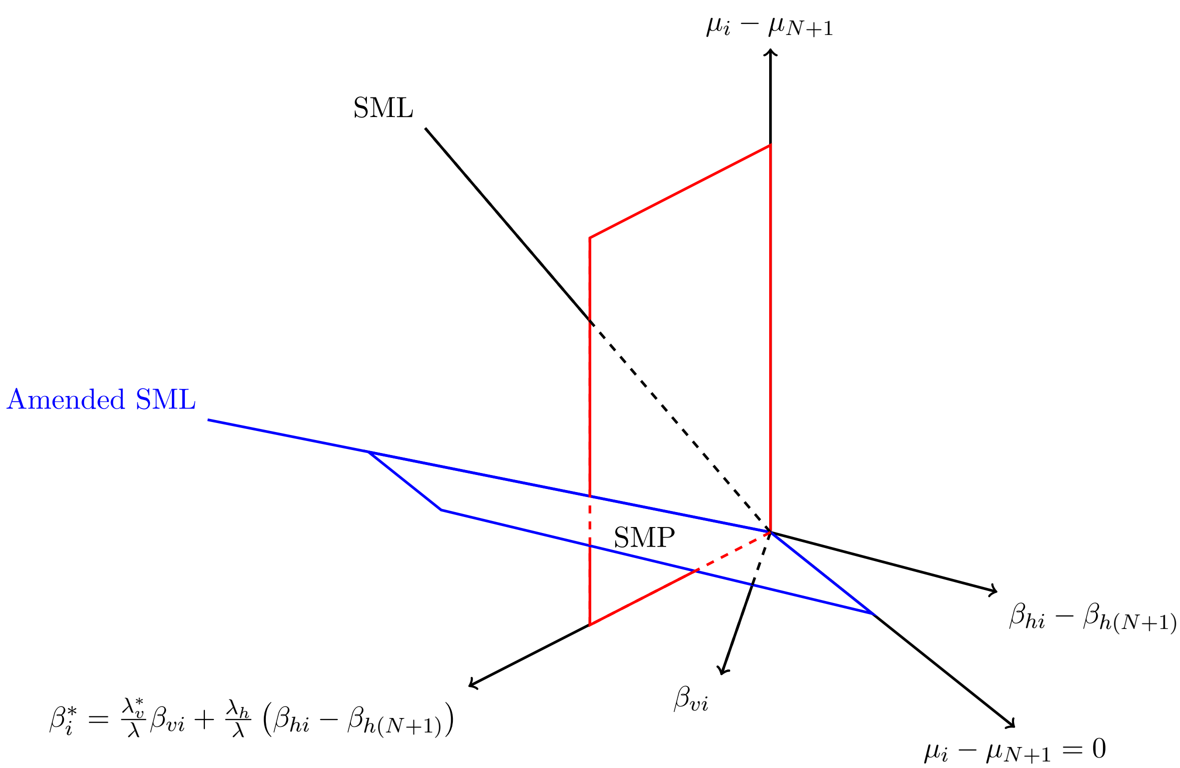
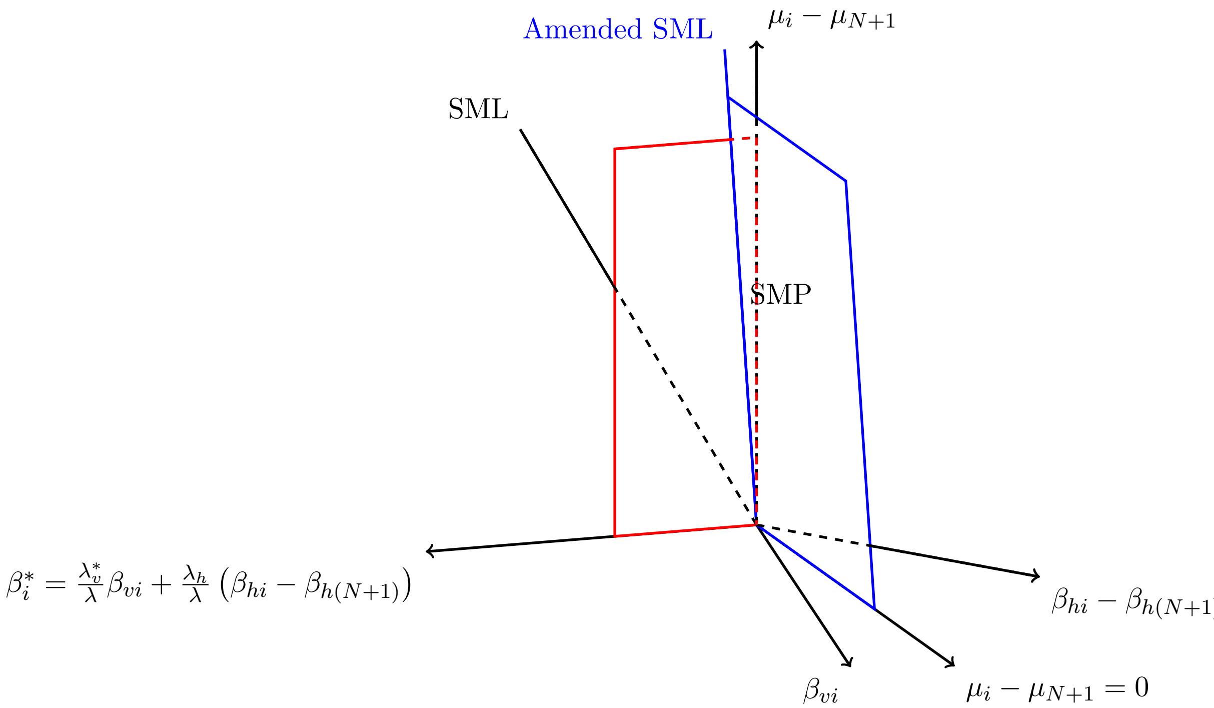


Figure 3: Security Market Plane (SMP): $\lambda_v^* = 0.5$, $\lambda_h = -1$, $\iota_m = 0.5$, $\lambda = 0.5$



4.2 Pricing of Idiosyncratic Risk

Now let us demonstrate that idiosyncratic risk is priced in our general RQE-CAPM. Without loss of generality, we consider the case where a risk-free asset is present. Following the steps (34)-(36), it is straightforward to prove that the premium λ_v^{*I} on the idiosyncratic risk β_{vi}^I is

$$\lambda_v^{*I} = \lambda_v^* \, \beta_{vi}^I = \frac{\iota_m}{\tau_m}\sigma_{e_i}^2,$$

and the premium λ_h^I on the idiosyncratic risk β_{hi}^I remains the same

$$\lambda_h^I = \lambda_h \, \beta_{hi}^I = -\frac{\sigma_{e_i}^2}{\tau_m}.$$

These two premiums do not sum to zero

$$\lambda_v^{*I} + \lambda_h^I = (\iota_m - 1)\frac{\sigma_{e_i}^2}{\tau_m}.$$

Idiosyncratic risk is therefore priced when the societal ratio, ι_m, is not equal to 1. Specifically, it is negatively (positively) priced when $\iota_m < 1$ ($\iota_m > 1$).

The intuition behind this result is as follows. When the societal ratio, ι_m, is lower than 1, the societal appetite to reduce risk through correlation diversification is higher than that to reduce risk through concentration on assets with low variance. This implies that the demand of assets with low variance, all things equal, is lower than those with high variance. As a consequence assets with high potential correlation diversification are overpriced, which would then earn then high subsequent returns, while assets with low potential correlation diversification are underpriced, earning then high subsequent returns. Since assets with high potential correlation diversification are those with high variance and low correlations, assets with high variance have higher prices than those with low variance, which then translates into subsequent higher returns. However, assets with low variance command lower prices, which means higher subsequent returns. Expected returns are therefore decreasing functions of asset variances and idiosyncratic risks.

Take the second case now, with $\iota_m > 1$. This means that the societal ratio is greater than 1. As such, the societal appetite to reduce risk through correlation diversification is lower than that to reduce risk through concentration on assets with low variance. This implies that the demand of assets with low variance, all things equal, is higher than those with high variance. As a consequence, assets with low variance are overpriced, which *expost* translates into lower returns for these assets. Conversely, assets with high variance are underpriced, which leads to subsequent higher returns. Expected returns are therefore increasing functions of asset variances and idiosyncratic risks.

Our general RQE-CAPM offers therefore a unifying way of thinking about the pricing of idiosyncratic risk and can predict a positive, a negative, or a non relationship between expected return and idiosyncratic risk. It is therefore relevant for the puzzle whereby idiosyncratic risk is negatively priced, as documented in the studies by Ang et al. (2006), Ang et al. (2009), Guo and Savickas (2010) and Boyer et al. (2010). According to Hou and Loh (2016), the puzzle remains largely unexplained. Only explanations based on investors' lottery preferences and market frictions show some promise. Our general RQE-CAPM offers interesting new preference-based explanations for this puzzle, where investors' affinity for high idiosyncratic volatility stems from a preference for correlation diversification and a desire to hold a more correlation diversified portfolio, rather than an inclination towards speculation or gambling, as in other contributions (Bali et al., 2011; Barberis and Huang, 2008; Boyer et al., 2010; Han and Kumar, 2013; Mitton and Vorkink, 2007).

4.3 Cross-Sectional Implications

We end by showing that our general RQE-CAPM also offers cross-sectional implications that can be tested directly. Consider an alternative formulation of the utility function (37) replacing $H(\mathbf{w})$ from (9)

$$U(\mathbf{w}_k) = \mathbf{w}_k^\top \boldsymbol{\mu} + \left(\frac{1}{\tau_k} - \frac{1}{\varsigma_k}\right) \mathbf{w}_k^\top \boldsymbol{\sigma}^2 - \frac{1}{\tau_k} \sigma^2(\mathbf{w}_k). \tag{46}$$

Following the same derivation steps of (19) and (41), we obtain the following testable equilibrium relation

$$\mu_i = \nu_m^{**} + \left(\frac{\iota_m - 1}{\tau_m}\right) \sigma_i^2 + \left(2\frac{\sigma_m^2}{\tau_m}\right) \beta_i, \ \ \forall\, i = 1, ..., N, \tag{47}$$

where $\nu_m^{**} = \frac{\sum_{k=1}^K W_k \tau_k \nu_k^{**}}{\tau_m}$ is the societal marginal utility of wealth. In the absence of a risk-free rate, ν_m^{**} can also be interpreted as the return, $\mu_{zzz} = \sum_{i=1}^N w_{(zzz)i}\mu_i$, on a portfolio $\mathbf{w}_{zzz}$ such that $\beta_{v(zzz)} = \sum_{i=1}^N w_{(zzz)i}\sigma_i^2 = 0$ and $\beta_{zzz} = \sum_{i=1}^N w_{zzz,i}\beta_i = 0$. It is straightforward to verify that a portfolio $\mathbf{w}_{zzz}$ always exists when $N > 3$. However, in the presence of a risk-free asset, $\nu_m^{**} = \mu_{N+1}$, because $\sigma_{N+1}^2 = \beta_{N+1} = 0$.

Applying the relation (47) to the portfolio $\mathbf{w}_z$ and the market portfolio, one can deduce that

$$\frac{\iota_m - 1}{\tau_m} = \frac{\mu_z - \nu_m^{**}}{\bar{\sigma}_z^2}, \tag{48}$$

$$\mu_m = \nu_m^{**} + \left(\frac{\iota_m - 1}{\tau_m}\right) \bar{\sigma}_m^2 + \left(2\frac{\sigma_m^2}{\tau_m}\right), \tag{49}$$

respectively. Combining (48) and (49), we obtain

$$2\frac{\sigma_m^2}{\tau_m} = (\mu_m - \nu_m^{**}) - (\mu_z - \nu_m^{**})\,\frac{\bar{\sigma}_m^2}{\bar{\sigma}_z^2}. \tag{50}$$

Substituting (48) and (50) into (47), we obtain

$$\mu_i = \nu_m^{**} + \left(\frac{\mu_z - \nu_m^{**}}{\bar{\sigma}_z^2}\right) \sigma_i^2 + \left((\mu_m - \nu_m^{**}) - (\mu_z - \nu_m^{**})\,\frac{\bar{\sigma}_m^2}{\bar{\sigma}_z^2}\right) \beta_i, \ \ \forall\, i = 1, ..., N. \tag{51}$$

An alternative formulation of (51) replacing σ_i^2 from (24) is

$$\mu_i = \nu_m^{**} + \left(\frac{\mu_z - \nu_m^{**}}{\bar{\sigma}_z^2}\right)\sigma_{e_i}^2 + \left((\mu_z - \nu_m^{**})\frac{\sigma_m^2}{\bar{\sigma}_z^2}\right)\beta_i^2 + \left((\mu_m - \nu_m^{**}) - (\mu_z - \nu_m^{**})\frac{\bar{\sigma}_m^2}{\bar{\sigma}_z^2}\right)\beta_i, \ \ \forall\, i = 1, ..., N. \tag{52}$$

Equations (51) and (52) are the testable versions of our general RQE-CAPM. In practice, they can be tested using the standard two pass regression of Lintner performed in Levy (1978), or Fama and MacBeth's (1973) two pass regression.

Several prior studies have already implicitly used the testable versions of our general RQE-CAPM to test the validity of the standard CAPM or the pricing of idiosyncratic risk. For example, see Levy (1978), Malkiel and Yexiao (2006), Lakonishok and Shapiro (1986) , Friend and Westerfield (1981), Lakonishok and Shapiro (1984), Carroll and Wei (1988), Cadsby (1992), Lehmann (1990), Amihud and Mendelson (1989) and Mehra et al. (2021), among others. The empirical tests contained in these studies can therefore be interpreted as empirical tests of our general RQE-CAPM. In conjunction, our general RQE-CAPM provides potential theoretical support for these empirical tests.

5 Conclusion

Sharpe (1964), Lintner (1965), Mossin (1966) and Treynor (1961) used the mean-variance utility function to derive the capital asset pricing model (CAPM), which has remained the most popular asset pricing model. Carmichael et al. (2015) introduces a novel representation of Markowitz's (1952) mean-variance utility function based on Rao's Quadratic Entropy (RQE) and this paper uses this representation to update the CAPM, both in the presence and in the absence of a risk-free asset. The resulting derivation, which we denote RQE-CAPM, provides new insights on the pricing mechanism at play in the standard CAPM and reveals the important role of agents' risk aversion and correlation diversification risk preference coefficients in the pricing of idiosyncratic risk. Specifically, it predicts that an asset's total risk (measured by its variance) is priced positively, while its correlation diversification risk is priced negatively (or taxed). Idiosyncratic risk is therefore priced twice: positively because it is a risk and negatively because it is a correlation diversification risk. These opposing effects cancel each other out in the standard CAPM because agents' total risk aversion and correlation diversification risk preference coefficients are equal. As a result, idiosyncratic risk is not priced in the standard CAPM, not only because the market portfolio is *law of large numbers* diversified, but also because agents' total risk aversion and correlation diversification risk preference coefficients are equal.

Next, the paper answers the following question: should idiosyncratic risk be priced when agents' risk aversion and correlation diversification risk preference coefficients are not equal? To do so a general RQE-CAPM, both in the presence and in the absence of a risk-free asset, in which we have relaxed the implicit assumption that agents' risk aversion and correlation diversification risk preference coefficients are equal, is developed. This general RQE-CAPM shows that the pricing of idiosyncratic risk depends upon the societal ratio of total risk aversion coefficient to correlation diversification risk preference aversion coefficient: idiosyncratic risk is priced positively when the ratio is greater than 1, negatively when the ratio is lower than 1, and is not priced when the ratio is equal to 1. Our general RQE-CAPM offers therefore a unifying way of thinking about the pricing of idiosyncratic risk. As a corollary, it constitutes a new potential theoretical preference-based explanation for the well-documented idiosyncratic risk puzzle.

The paper concludes with a discussion of the cross-sectional restrictions on expected returns implied by our general RQE-CAPM. It points out that several studies in the empirical literature have implicitly used these restrictions in an *ad hoc* manner to test the validity of the standard CAPM or the pricing of idiosyncratic risk. The analysis of this paper can therefore be seen as offering a theoretical underpinning for the empirical tests performed in these studies.

Further research could attempt to analyze asset pricing considering the RQE representation of Markowitz's (1952) mean-variance utility function (14) but with different dissimilarity matrices. Another line of research could seek to study the implications of the utility function (14) in terms of Markowitz's (1952) mean-variance portfolios stability and diversification.

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