2015-02

Unifying Portfolio Diversification Measures Using Rao's Quadratic Entropy

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mai / may 2015

Centre de recherche sur les risques
les enjeux économiques et les politiques publiques

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Abstract

This paper extends the use of Rao(1982b)'s Quadratic Entropy (RQE) to modern portfolio theory. It argues that the RQE of a portfolio is a valid, flexible and unifying approach to measuring portfolio diversification. The paper demonstrates that portfolio's RQE can encompass most existing measures, such as the portfolio variance, the diversification ratio, the normalized portfolio variance, the diversification return or excess growth rates, the Gini-Simpson indices, the return gaps, Markowitz's utility function and Bouchaud's general free utility. The paper also shows that assets selected under RQE can protect portfolios from mass destruction (systemic risk) and an empirical illustration suggests that this protection is substantial.

Keywords : Portfolio Diversification, Rao's Quadratic Entropy, Diversification Return, Diversification Ratio, Portfolio Variance Normalized, Gini-Simpson Index, Markowitz's Utility Function, Bouchaud's General Free Utility

JEL Classification : G11

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1 Introduction

Rao(1982b)'s Quadratic Entropy (RQE)[1] is a general approach to measuring diversity which has been used extensively in fields such as statistics and ecology. In statistics, Rao (1982a,b) and Nayak (1986a,b) used RQE to generalize the analysis of the variance; in ecology, several studies used RQE as a biodiversity measure (Champely and Chessel, 2002; Pavoine, 2012; Pavoine and Bonsall, 2009; Pavoine et al., 2005; Ricotta and Szeidl, 2006; Zhao and Naik, 2012). RQE has also been used by Stirling (2010) to study energy policy and define a general framework for analysing energy diversity, while Nayak and Gastwirth (1989) have also used RQE to analyze the relative effects of factors such as age, sex, and education on the distribution of income.

Extending its use to portfolio theory, this paper proposes that RQE becomes the organizing principle to measure portfolio diversification. To this end, we show that the RQE of a portfolio (portfolio RQE) has four characteristics that make it particularly appropriate for that function: (i) it meets ex-ante desirable properties of diversification; (ii) it unifies several portfolio diversification measures and utility functions that have been analyzed in the literature, giving them a common, theoretically-grounded interpretation as special cases or function of RQE; (iii) it provides a flexible but formal approach for fund managers to develop new, diversified portfolios and (iv) it can provide protection from systemic as well as idiosyncratic risk.

Diversification is at the core of portfolio selection in Modern Portfolio Theory (MPT), yet there exists no formal definition for this concept and, as a result, no unique measure to quantify the degree to which a given portfolio is diversified. Developing an "ideal "measure of portfolio diversification is therefore an active research area in investment management and we show that portfolio RQE possesses necessary properties to be this measure.

The remainder of this paper is organized as follows. Section 2 reviews the existing literature on measures of portfolio diversification and suggests that it lacks a formal, unifying measure. Section 3 defines portfolio RQE, describes how to select the optimal portfolio in that context, and presents some particular cases. Section 4 and 5 discuss the main properties of portfolio RQE and portfolios optimized according to this criterion. Section 6 provides an empirical

[1] RQE is also referred to as Diversity Coefficient (Rao, 1982b) or Quadratic Entropy (Rao and Nayak, 1985).

illustration of the capacity of portfolio RQE to protect from mass destruction (systemic risk). Section 7 concludes.

2 Portfolio Diversification Measures

The first mathematical formalization of diversification in portfolio selection analysis dates back to the mean-variance model of Markowitz (1952). Although it analyzes the idea of portfolio diversification, the model does not provide a specific measure of portfolio diversification. Thus, it does not answer the simple question of whether a specific portfolio is well diversified.

Since Markowitz (1952), several measures of diversification have been proposed in the literature, each based on a different criterion. Using the Capital Asset Pricing Model (CAPM), Evans and Archer (1968) suggest portfolio size. Sharpe (1972)'s measure is based on portfolio idiosyncratic risk. Fernholz and Shay (1982) introduce the excess growth rate, also known as diversification return (see Booth and Fama, 1992; Bouchey et al., 2012; Chambers and Zdanowicz, 2014; Qian, 2012; Willenbrock, 2011), while Woerheide and Persson (1993) appeals to the Gini-Simpson (GS) index of the portfolio. More recently, Statman and Scheid (2005) base their proposed measure on the return gaps (RG), Rudin and Morgan (2006) develop portfolio diversification indices (PDI) derived from principal component analysis (PCA), Choueifaty and Coignard (2008) recommend their diversification ratio (DR), Goetzmann et al. (2005) and Goetzmann and Kumar (2008) put forward their portfolio variance normalized while Meucci (2009) and Meucci et al. (2014) analyze measures based on the effective number of bets (ENB) using a PCA, a minimum torsion bets and Shannon entropy. Finally, Vermorken et al. (2012) have developed the diversification delta, a higher-moment measure for portfolio diversification using Shannon entropy.[2]

However, none of the above measures have proven totally satisfactory. For instance, portfolios based on Sharpe (1972)'s measure may be formed with assets having similar high betas, leaving them exposed to market risk. The GS index is useful only when zero information is available on the various assets and naive diversification (i.e. equal weights) is optimal. The DR has been criticized by Lee (2011) and Taliaferro (2012) for not being associated with a clear objective function so that a DR-maximizing portfolio "only has desirable properties

[2]For more details on portfolio diversification measures since Markowitz (1952), see Frahm and Wiechers (2011), Fragkiskos (2013),Pola (2013) and Carli et al. (2014).

by accident "(Taliaferro, 2012). The ENB has two major shortcomings : it does not distinguish between negative and positive correlation (and thus cannot incorporate the benefits arising from negative correlation) and it can only be computed if portfolio risk is measured by its variance or volatility, which are adequate measures of risk only in the unlikely case that asset returns are normally distributed (Embrechts et al., 1999).

Developing a unifying, theoretically-motivated approach to measure portfolio diversification is therefore an important research goal. This paper proposes portfolio RQE as such an approach. We show that doing so has four important advantages. First, portfolio RQE satisfies properties deemed desirable for a measure of portfolio diversification. In particular, it verifies i) Choueifaty et al. (2013)'s duplication invariance property, ii) Markowitz (1952)'s property that a portfolio of less dissimilar assets is likely to offer less diversification than one of more dissimilar assets, and iii) portfolio RQE can be decomposed across assets class and times periods, allowing it to be fully consistent with the old adage "don't put all your eggs in one basket "(see for example Markowitz et al., 2009, pp. 12). Second, portfolio RQE is an unifying approach. We prove that it is the core around which existing measures of portfolio diversification (the diversification ratio, the portfolio variance normalized, the diversification return or the excess growth rate, the Gini-Simpson index, the return gap) are built. Third, it provides fund managers with a flexible but formal approach for the construction of portfolios diversified according to a variety of characteristics. Fourth, portfolios optimized according to portfolio RQE (RQE portfolios or RQEP in short) offer protection both from asset-specific shocks (systematic risk) as well as mass destruction (systemic risk). Section 6 of the paper provides an empirical illustration of the capacity of RQEP to protect from mass destruction, by comparing the in-sample and out-sample performance of four RQEP relative to those of the most popular existing diversified portfolios. Using two performance measures (portfolio return during bear markets and portfolio market beta), we show that RQEP dominates existing measures.

3 Definition of Portfolio Rao's Quadratic Entropy and its Optimal Portfolios

We begin by reviewing the general formulation of RQE. Let Ω be a population of individuals. Suppose that each individual in Ω is characterized by a set of measurement X and denote by P the probability distribution function of X. Rao (1982b) defines the RQE of Ω as the average difference between two randomly drawn individuals from Ω:

$$H(P) = \int d(X_1, X_2) P(dX_1) P(dX_2), \tag{3.1}$$

where the non-negative, symmetric "dissimilarity "function $d(.,.)$ expresses the difference between two individuals from Ω. When X is a discrete random variable, $H(P)$ becomes

$$H(P) = \sum_{i,j}^{N} d_{ij} p_i \, p_j, \tag{3.2}$$

where $p_i = P(X = x_i)$, $\forall\, i = 1, ..., N$.

As can be noted, RQE differs from other entropy measures used in finance, such as Shannon entropy and Tsallis Entropy (Zhou et al., 2013), by the fact that it incorporates not only the relative abundance of individuals p_i, but also the intrinsic difference between individuals, d_{ij}. This is the source of its attraction and flexibility.

3.1 Portfolio RQE

We now extend the definition of RQE to portfolio selection, using the discrete version (3.2) because it is more naturally suited to portfolio selection. To do so, we must define the relevant population and its individuals, as well as specify the random variable X, its distribution probability function P, and the dissimilarity function, $d(.,.)$.

Consider an universe $U = \{A_i\}_{i=1}^{N}$ of N different assets, and denote by $w = (w_i)_{i=1}^{N}$ a specific long-only portfolio, where w_i is the weight of asset i in w, so that each portfolio w can be viewed as a population of individuals assets. Next, define the random variable X to take the finite values $1, ..., N$ (N assets) and its probability distribution $P(X = i) = w_i$, $\forall\, i$, so that it is associated to the random experiment whereby assets are randomly selected (with replacement) from portfolio w. Finally, specify $D = (d_{ij})_{i,j=1}^{N}$ as the dissimilarity function measuring the difference between any two assets of the portfolio, with D satisfying the following conditions : $d_{ij} \geq 0$, $\forall\, i, j = 1, ..., N$; $d_{ij} = d_{ji}$, $\forall\, i, j = 1, ..., N$; $d_{ii} = 0$, $\forall\, i = 1, ..., N$. For the purpose of this paper, we also consider D to be independent from X. Given a portfolio w and the dissimilarity matrix $D = (d_{ij})_{i,j=1}^{N}$, the RQE of a portfolio w (portfolio RQE) is defined as half of the average difference between two randomly drawn (with replacement) assets from w, as follows:

$$H_D(w) = \frac{1}{2} \sum_{i,j=1}^{N} d_{ij}\, w_i\, w_j. \tag{3.3}$$

Furthermore, if D is conditionally negative definite, H_D is concave and following Rao and Nayak (1985) $H_D(w)$ can be rewritten as

$$H_D(w) = \frac{1}{2} \sum_{i=1}^{N} w_i\, D_{H_D}\left(w^i, w\right) \tag{3.4}$$

where w^i is the i^{th} single-asset portfolio ($w^i_i = 1$ and $w^i_j = 0$, $j \neq i$) and

$$D_{H_D}\left(w^i, w\right) = 2\, H_D(w^i, w) - H_D(w^i) - H_D(w), \tag{3.5}$$

is a symmetric cross entropy associated to H_D and represents a dissimilarity on W measuring a difference between two portfolios, and its squared root an Euclidean distance, with $H_D(w^i, w) = w^\top D\, w^i$. This alternative definition facilitates the interpretation of $H_D(w)$: it is a measurement of the average difference between portfolio w and the single asset portfolios : the closer w is to the single portfolios, the less diversified it is and the smaller the index $H_D(w)$; conversely the further w is from the single portfolios, the more diversified it is and the higher is $H_D(w)$. Given two portfolios, portfolio managers will prefer that which has a high portfolio RQE as is less close to the single asset portfolios.

Another interpretation of portfolio RQE is in terms of the average degree of information concentration. Under this interpretation, d_{ij} measures the quantity of unshared information of assets i and j. Then, the less dissimilar assets are, the more concentrated is a portfolio in terms of information and the lower is portfolio diversification as measured by portfolio RQE. In the extreme case where all assets are perfectly similar ($d_{ij} = 0$, $\forall\, i, j$), the degree of diversification is nil and portfolio RQE is zero with $H_D(w) = 0$. Since portfolio RQE also takes the value zero for a single asset portfolio, portfolio RQE can be interpreted as a measure of information concentration and fund managers will prefer portfolios that have high portfolio RQE. Furthermore, when d_{ij} are normalized in the range $[0, 1]$, the effective number of independent risk factors in a portfolio w, similar to Bouchaud and Potters (2000) and following Ricotta and Szeidl (2009), can be measured by

$$N_{eff}^{RQE}(w) = \frac{1}{1 - 2\,H_D(w)}$$

As we can see, N_{eff}^{RQE} generalizes Bouchaud and Potters (2000, p. 111)'s measure since H_D generalizes the Gini-Simpson index as we will be seen in Section 3.3.

Remark 3.1. *A formal proof that portfolio RQE is a measure of portfolio concentration in terms of information can be provided using Theorem 1 of Bavaud (2010).*

Remark 3.2. *Portfolio managers can make the choice between two portfolios using a standard two independent samples t-test. Indeed, consider the random experiment which consists of randomly selecting with replacement two assets from a portfolio. Define Z as the discrete random variable equal to the dissimilarity between the two assets drawn, so that Z takes a finite number of values d_{ij}, $i, j = 1, ..., N$. The probability distribution of Z is $P(Z = d_{ij}) = w_i\, w_j$. One can show that portfolio RQE is half of the mean of Z. Comparing two portfolio RQE is therefore similar to comparing the mean of two random variables, and this comparison can be conducted using the test statistic*

$$T = \frac{\mu_Z(w) - \mu_{\boldsymbol{Z}}(\boldsymbol{w})}{\sqrt{\frac{S_Z^2(w)}{N_Z} + \frac{S_{\boldsymbol{Z}}^2(\boldsymbol{w})}{N_{\boldsymbol{Z}}}}}$$

where $N_Z = 2\frac{N_w!}{2!(N_w-2)!}$ is the number of values Z can take with N_w portfolio w size, $N_{\boldsymbol{Z}} = 2\frac{N_{\boldsymbol{w}}!}{2!(N_{\boldsymbol{w}}-2)!}$ the number of values $\boldsymbol{Z}$ can take with $N_{\boldsymbol{w}}$ portfolio $\boldsymbol{w}$ size, $\mu_Z(w) = 2\,H_D(w)$ a mean of Z, $\mu_{\boldsymbol{Z}}(\boldsymbol{w}) = 2\,H_{\boldsymbol{D}}(\boldsymbol{w})$ a mean of $\boldsymbol{Z}$, $S_Z^2(w) = 2\,H_{D^2}(w) - (2\,H_D(w))^2$ a variance of Z and $S_{\boldsymbol{Z}}^2(\boldsymbol{w}) = 2\,H_{\boldsymbol{D}^2}(\boldsymbol{w}) - (2\,H_{\boldsymbol{D}}(\boldsymbol{w}))^2$. The distribution of T is approximated as an ordinary Student's t distribution with the degrees of freedom calculated using

$$df = \frac{\left(\frac{S_Z^2(w)}{N_Z} + \frac{S_{\boldsymbol{Z}}^2(\boldsymbol{w})}{N_{\boldsymbol{Z}}}\right)^2}{\frac{S_Z^4(w)}{N_Z^2\,(N_Z-1)} + \frac{S_{\boldsymbol{Z}}^4(\boldsymbol{w})}{N_{\boldsymbol{Z}}^2\,(N_{\boldsymbol{Z}}-1)}}$$

This test is applicable only if $N_Z,\ N_{\boldsymbol{Z}} \geq 30$. That is the case when $N_w,\ N_{\boldsymbol{w}} \geq 6$, which is generally the case.

3.2 RQE Portfolios

Portfolio RQE optimal portfolios (RQE portfolios or RQEP in short) is therefore defined as

$$w^{RQE} \in \underset{w \in W}{\arg\operatorname{Max}}\, H_D(w), \tag{3.6}$$

i.e., the portfolios that minimize the information concentration, or that maximize the effective number of independent risk factors, with W is the set of long-only portfolios.

Rao (1982b), Pavoine et al. (2005) and Pavoine and Bonsall (2009) discuss the conditions under which problem (3.6) has a solution. They show that when H_D is concave, a global maximum exists

$$w^{RQE} = \frac{D^{-1}\mathbf{1}}{\mathbf{1}^\top D^{-1}\mathbf{1}}$$

where the solution w^{RQE} is interior, $\mathbf{1}$ is a vector column of one . Pavoine et al. (2005) shows that w^{RQE} is unique when D is ultrametric. When D is not conditionally negative definite, the problem (3.6) still has a solution.

Remark 3.3. *The Definitions (3.3) and (3.6) can be extended to the case of short sales at the price of some additional complexities.*

As we can observe from (3.6), the optimal RQE portfolio is solely determined by asset dissimilarity; in particular, asset i's weight w_i^{RQE} is a strictly increasing function of the asset's dissimilarity contribution as shown in Proposition 3.1.

Proposition 3.1. *Let two assets i and j held in the RQEP i.e. $w_i^{RQE}, w_j^{RQE} > 0$. Then*

1. $DC_i\left(w^{RQE}\right) < DC_j\left(w^{RQE}\right) \iff w_i^{RQE} < w_j^{RQE}$.
2. $DC_i\left(w^{RQE}\right) = DC_j\left(w^{RQE}\right) \iff w_i^{RQE} = w_j^{RQE}$.

where

$$DC_i\left(w^{RQE}\right) = \sum_{k=1}^{N_{w^{RQE}}} d_{ik}$$

is the dissimilarity contribution of an asset i held in the RQEP, with $N_{w^{RQE}}$ the number of assets held in the RQEP.

Proof of Proposition 3.1. See Appendix A.1. ■

Proposition 3.1 implies that RQEP spreads eggs by putting more eggs in more dissimilar baskets and fewer eggs in less dissimilar baskets. As a result, holding RQEP can help reduce losses during bear markets or financial crises, since the more dissimilar are the baskets, the less is the probability that they do poorly at the same time in the same proportion.

Remark 3.4. *Proposition 3.1 implies that RQEP avoids overweight in terms of information.*

3.3 Particular Cases

Equation (3.3) constitutes a general definition of portfolio RQE but in practice, one needs to specify the dissimilarity matrix D. We present four alternative specifications of D, of which three are taken within the perimeter of the mean-variance paradigm.

3.3.1 Gini-Simpson Index

Suppose that D is specified such as

$$d_{ij} = d\,(1 - \delta_{ij}), \tag{3.7}$$

where δ_{ij} is Kronecker's delta and d is a strictly positive constant. In that case, portfolio RQE is equivalent to the Gini-Simpson index, a weight-based measure commonly used in the literature (Cazalet et al., 2014; Woerheide and Persson, 1993; Yanou, 2010). As a result, portfolio RQE generalizes the Gini-Simpson index and it resolves some of its shortcomings as a measure of portfolio diversification.[3]

Now, suppose that $d = 2\,\sigma_0^2\,(1 - \rho_0)$, $N = 2$ and $w_1 = w_2 = \frac{\mathbf{1}}{2}$. Then, $H_D(w)$ becomes

$$H_D\left(\frac{\mathbf{1}}{N}\right) = \frac{1}{2}\,\sigma_0^2(1 - \rho_0), \tag{3.8}$$

[3] Frahm and Wiechers (2011) point out that the Gini-Simpson's main problems is related to the axiomatization of concentration measures, especially in the axioms of symmetry and monotonicity. For example, the axiom of symmetry implies that "the portfolio weights can be exchanged without any alteration of the degree of diversification". This implies exchangeability of the distribution of the asset returns without any alteration of the degree of diversification, a result not supported by the GS index.

where σ_0 is asset volatility and ρ_0 its correlation. Taking the square root of (3.8):

$$\sqrt{H_D\left(\frac{\mathbf{1}}{N}\right)} = \sigma_0\sqrt{\frac{1-\rho_0}{2}}. \tag{3.9}$$

The right side of the above equality is the return gaps (RG), a portfolio diversification measure introduced under the name of "dispersion"by Statman and Scheid (2005) as an alternative to the use of correlation, and later expanded upon by Statman and Scheid (2008). Equation (3.9) thus shows that portfolio RQE also generalizes the RG.

3.3.2 Diversification Return or Excess Growth Rate

Suppose that D is based on the asset returns covariance matrix such that

$$d_{ij} = \sigma_i^2 + \sigma_j^2 - 2\sigma_{ij}, \tag{3.10}$$

where σ_i is the variance of asset i and σ_{ij} is the covariance between asset i and j. In that case,

$$H_D(w) = \frac{1}{2}\sum_{i,j=1}^{N}(\sigma_i^2 + \sigma_j^2 - 2\sigma_{ij})w_i w_j, \tag{3.11}$$

and one can show that portfolio RQE is equivalent to diversification return (Booth and Fama, 1992; Willenbrock, 2011) or the excess growth rate (Fernholz, 2010; Fernholz and Shay, 1982).[4] Portfolio RQE therefore also generalizes the diversification return and illustrates that this measure takes its source in RQE diversification in the spirit of Markowitz (1952)'s idea, as opposed to diversification as traditionally defined in the CAPM.[5]

[4]The diversification return (Dr) is defined by Booth and Fama (1992) as the difference between the portfolio compound return and the weighted average asset compound return. An equivalent concept was previously introduced under the name of excess growth rate by Fernholz and Shay (1982) and it now commonly used in literature (for a detailed review see Chambers and Zdanowicz, 2014) . The mathematical formula of the diversification return is

$$Dr(w) = \frac{1}{2}\left(w^\top\sigma^2 - w^\top\,\Sigma\, w\right).$$

Booth and Fama (1992) argue that the diversification return is assured only if the investor maintains relatively fixed asset weights and its may be lost by engaging in active management. Willenbrock (2011), Qian (2012) and Bouchey et al. (2012) argue that it is due to rebalancing and diversification. Chambers and Zdanowicz (2014) argue that "not diversification as traditionally defined, that generates diversification return", and diversification return provided increased expected value only if prices are mean-reverting.

[5]See Chambers and Zdanowicz (2014) for a discussion.

3.3.3 Portfolio variance

Take particular case 3.3.2 and suppose further that assets have equal volatility, normalized to one. As a result, D is based on the matrix of asset returns correlation $\rho = (\rho_{ij})_{i,j=1}^{N}$

$$d_{ij} = 2(1 - \rho_{ij}), \tag{3.12}$$

so that portfolio RQE becomes

$$H_D(w) = \sum_{i,j=1}^{N} (1 - \rho_{ij}) w_i w_j. \tag{3.13}$$

As we can observe, H_D is a decreasing function in ρ_{ij} and portfolio RQE diversification vanishes if assets are perfectly correlated. Thus, portfolio RQE embodies the intuition that low correlation implies a high degree of diversification.

Finally, one can show that portfolio RQE is equivalent to portfolio variance when the assets' volatilities are identical. In that case portfolio variance belongs to the larger family of portfolio RQE and diversify under RQE coincides exactly with risk reduction, where risk is measured by portfolio variance. Under the same conditions, one can also show that portfolio RQE is equivalent to the diversification ratio (DR) and the portfolio variance normalized (NV).

3.3.4 Factor Diversification

The particular cases discussed so far are based only on D taken within the perimeter of the mean-variance paradigm. However, D can be made to depend on various other asset characteristics, such as higher moments of assets returns, rank correlations, copula-based dependences, Granger causality, linear partial correlation, tail dependences, asset liquidity characteristics, etc, individually or simultaneously. Here, we present an example of specification of D not taken within the perimeter of the mean-variance paradigm.

Consider the multifactor model for asset returns

$$R_{it} = \alpha_i + \sum_{k=1}^{K} \beta_{ki} f_{kt} + \epsilon_{it}$$

where R_{it} is the return of asset i in time period t, f_{kt} is the k^{th} common factor, β_{ki} is the factor loading for asset i on the k^{th} factor and ϵ_{it} is the asset specific factor. Using standard assumptions (error terms ϵ_{it} are serially and contemporaneously uncorrelated across assets as well as uncorrelated with the common factors, f_{kt}, which one themselves are orthogonal) the matrix D can be specified as

$$D = (1-\theta)\, D^{CF} + \theta\, D^{SF},$$

with $d_{ij}^{CF} = \sum_{k=1}^{K}(\beta_{ki} - \beta_{kj})^2$ and d_{ij}^{SF} specified as (3.12). The corresponding portfolio RQE is

$$H_D(w) = (1-\theta)\, H_{D^{CF}}(w) + \theta\, H_{D^{SF}}(w),$$

where $H_{D^{CF}}(w)$ measures diversification across factors (f_{kt}) and betas. It can be used to construct a well-diversified portfolio free from risk if there is no essential factor risk.[6] For its part, $H_{D^{SF}}(w)$ measures the diversification across the specific factor ϵ_{it}. The parameter $\theta \in [0,1]$ can be interpreted as the investor's aversion toward specific risk : the higher is θ, the greater is aversion towards specific risk and the lower it is towards systematic risk.

3.4 Summary

Three main conclusions can be drawn from the above discussion. First, portfolio RQE is unambiguously a valid diversification measure. Second, portfolio RQE is a unifying approach that generalizes the Gini-Simpson Index, the diversification return or excess growth rate, as well as portfolio variance, diversification ratio, portfolio variance normalized and return gaps under some conditions. Third, portfolio RQE provides a flexible but formal approach to develop diversified portfolios according to any kind of assets characteristics. This is a major advantage of portfolio RQE, not present in existing measures that allows it to "fully describe portfolio diversification contrary to what is believed in literature. "[7]

[6] See Ingersoll (1984) for details concerning the construction of a well-diversified portfolio free from risk.

[7] In the literature (Pola, 2014a,b), it believes that a single measure can not fully describe portfolio diversification, since portfolio diversification should be addressed from many angles.

4 Proprieties of Portfolio RQE

This section reviews some proprieties of portfolio RQE that are deemed desirable for a measure of portfolio diversification.

4.1 Degeneracy relative to portfolio size

Property 1 (Degeneracy relative to portfolio size). *A single asset portfolio must have the lowest diversification degree.*

The desirability of Property 1 comes from the usual qualitative definition of portfolio diversification: "don't put all your eggs in one basket ", an adage which is violated when Property 1 is not verified. The following proposition establishes that portfolio RQE verifies this property.

Proposition 4.1. *Let w be a portfolio and w^i be the i^{th} single asset portfolio i.e $w_i^i = 1$ and $w_j^i = 0\, j \neq i$. Then*

$$H_D(w) \geq H_D(w^i) = 0, \;\; \forall\, i = 1, ..., N.$$

Proof of Proposition 4.1. See Appendix A.2. ■

Example of portfolio diversification measures that do not verify this property include portfolio variance and the portfolio diversification index (PDI). By contrast, measures that do verify it include the portfolio size, the Gini-Simpson (GS) index, the Shannon index and the diversification ratio (DR).

4.2 Degeneracy relative to dissimilarity

Property 2 (Degeneracy relative to dissimilarity). *A portfolio formed solely with perfectly similar assets must have the lowest diversification degree.*

Once again, the desirability of this property arises from the adage "don't put all your eggs in one basket", and not respecting Property 2 violates the adage. Proposition 4.2 shows that portfolio RQE verifies this property.

Proposition 4.2. *Consider a universe U of assets formed solely with the same assets ($U = \{A_i\}_{i=1}^N$ such as $A_i = A, \forall\, i = 1, ..., N$) and w a portfolio. Then*

$$H_D(w) = 0.$$

Proof of Proposition 4.2. See Appendix A.3.■

From Proposition 4.2, we can deduce that a portfolio formed solely with the same assets is equivalent to a single asset portfolio. Therefore, portfolio RQE is unbiased towards asset total multi-representation or asset total duplication.

Examples of portfolio diversification measures that do not verify this property include portfolio size, the GS index and the Shannon index. By contrast, measures that do verify it include the DR with dissimilarity equal to (3.12).

4.3 Duplication Invariance

Property 3 (Duplication Invariance (Choueifaty et al., 2013)). *Consider a universe where an asset is duplicated (for example, due to multiple listings of the same asset). An unbiased portfolio construction process should produce the same portfolio, regardless of whether the asset was duplicated.*

The intuition behind this property, proposed by Choueifaty et al. (2013) as desirable for diversification, is as follows. Consider a universe $U = \{A, B\}$ of two assets. From U, derive a new universe $\boldsymbol{U} = \{A, A, B\}$ where asset A is duplicated. The duplicate invariant stipulates that the universes U and $\boldsymbol{U}$ must have the same diversification degree, and the weight of asset A of U must be equal to the sum of those of assets A of $\boldsymbol{U}$. The following Proposition shows that portfolio RQE verifies this property.

Proposition 4.3. *Let $U = \{A_i\}_{i=1}^N$ a universe of N assets and $\boldsymbol{U} = U \cup \{B\}$ a new universe of $N+1$ assets such that asset B is a duplication of an asset A_k, $k \in \{1, ..., N\}$. Let w^{RQE} and $\boldsymbol{w^{RQE}}$ denote the weights of RQEP associated to U and $\boldsymbol{U}$, respectively. Then*

$$H_D\left(w^{RQE}\right) = H_{\boldsymbol{D}}\left(\boldsymbol{w^{RQE}}\right)$$

and

$$w_{A_i}^{RQE} = \boldsymbol{w}_{\boldsymbol{A_i}}^{\boldsymbol{RQE}},\ i \neq k \text{ and } w_{A_k}^{RQE} = \boldsymbol{w}_{\boldsymbol{A_k}}^{\boldsymbol{RQE}} + \boldsymbol{w}_{\boldsymbol{B}}^{\boldsymbol{RQE}}.$$

Proof of Proposition 4.3. See Appendix A.4. ■

Examples of portfolio diversification measures that do not verify this property include portfolio size, the GS index, the Shannon index and the effective number of bets (ENB) (see Pola, 2014a,b, for details). By contrast, the DR and the portfolio variance verify Property 3 (see Choueifaty et al., 2013, for details).

Remark 4.1. *Property 3 implies Property 2.*

4.4 Non decreasing in portfolio size

Property 4 (Non decreasing in portfolio size). *An increase in portfolio size does not decrease the degree of portfolio diversification.*

This property is seen desirable by Markowitz (1952, p. 89): "The adequacy of diversification is not thought by investors to depend solely on the number of different securities held. "The following shows that portfolio RQE verifies this property.

Proposition 4.4. *Let U a universe of N assets and $\boldsymbol{U}$ a universe of $N+1$ assets, derived from U by increasing the size of U from N to $N+1$. Then*

$$H_{\boldsymbol{D}}\left(\boldsymbol{w}^{\boldsymbol{RQE}}\right) \geq H_D\left(w^{RQE}\right).$$

Proof of Proposition 4.4. See Appendix A.5.■

Examples of portfolio diversification measures that do not verify this property include portfolio size, the GS index and the Shannon index, while the DR, the PDI and the portfolio variance verify it.

Remark 4.2. *Portfolio size uniquely determines portfolio RQE if and only if dissimilarity between assets is constant, i.e. $d_{ij} = d, \forall\ i, j = 1, ..., N$ with d a constant. In that case, it is straightforward to show that RQEP coincides with the equally-weighted portfolio and*

$H_D\left(\frac{\mathbf{1}}{N}\right)$ *is the following increasing function of portfolio size:*

$$H_D\left(\frac{\mathbf{1}}{N}\right) = \frac{N-1}{2\,N}d.$$

4.5 Non decreasing in dissimilarity

Property 5 (Non decreasing in dissimilarity). *A portfolio of less dissimilar assets is likely to offer less diversification than one of more dissimilar assets.*

This property is seen desirable by:

1. Markowitz (1952, p. 89): "A portfolio with sixty different railway securities, for example, would not be as well diversified as the same size portfolio with some railroad, some public utility, mining, various sort of manufacturing, etc. The reason is that it is generally more likely for firms within the same industry to do poorly at the same time than for firms in dissimilar industries. ";

2. Sharpe (1972, p. 75): "... For example, a portfolio of ten chemical securities is likely to offer less effective diversification than one of ten securities, each from a different industry. This type of difference is difficult to capture in a simple formula. ";

3. Klemkosky and Martin (1975, p. 153), but in the context of single factor model: "... the levels of diversification achieved for high versus low beta portfolios for a given portfolio size were significantly different with high beta portfolios requiring a substantially larger number of securities to achieve the same level of diversification as a low beta portfolio. ".

Portfolio RQE easily verifies this property via the dissimilarity matrix D. For illustrating purposes, consider the simplest case where w is a naive portfolio ($w = \mathbf{1}/N$), and portfolio RQE is an increasing function of portfolio total dissimilarity, $DT = \sum_{i,j=1}^{N} d_{ij}$. More dissimilar portfolios (higher value of DT) are therefore more likely to offer diversification than less dissimilar portfolio (lower value of DT).

Examples of portfolio diversification measures that do not verify this property include portfolio size, the GS index, the Shannon index and the DR.

4.6 Portfolio RQE Decomposition

In addition, portfolio RQE also has the property to be decomposed across asset class and across time periods.

4.6.1 Asset Class Decomposition.

Suppose there are K asset classes and each class k has N_k single assets. Suppose further that portfolio RQE is concave. Then, following Rao (1982a), portfolio RQE can be decomposed into the sum of the weighted average of the dissimilarity within asset classes and the weighted average of the dissimilarity between all pairs of asset classes

$$H_D(w) = \frac{1}{2}\sum_{k=1}^{K} n_k\, H_{D,k}(w_k) + \frac{1}{2}\sum_{k=1}^{K}\sum_{l=1}^{K} n_k\, n_l D_{H_D}(w_k, w_l), \tag{4.1}$$

where n_k is the share of capital invested on the asset class k and $w_{k,i}$ is the share of N_k invested on the asset i of the asset class k ($w_i = n_k\, w_{k,i}$). Portfolio RQE therefore incorporates a trade-off between diversification within asset classes (the first term), and diversification between all pairs of asset classes (the second term).

Using (4.1), one can measure diversification within asset classes as follows :

$$H_{D,\text{within}} \equiv \frac{1}{2}\frac{\sum_{k=1}^{K} n_k\, H_{D,k}(w_k)}{H_D(w)},$$

and diversification between all pairs of asset classes as follows

$$H_{D,\text{between}} \equiv \frac{1}{2}\frac{\sum_{k=1}^{K}\sum_{l=1}^{K} n_k\, n_l\, D_{H_D}(w_k, w_l)}{H_D(w)}$$

with the contribution of an asset class k being

$$RQEC_k \equiv \frac{1}{2}\frac{n_k\, H_{D,k}(w_k) + \sum_{l=1}^{K} n_k\, n_l\, D_{H_D}(w_k, w_l)}{H_D(w)}.$$

Decomposition (4.1) can also be used to focus on diversification between all pairs of asset classes by setting portfolio w_k constant. To illustrate, assume that D is specified by (3.10) and set $w_k = \frac{\mathbf{1}}{N}, \forall\, k = 1, ..., K$. One can show that

$$H_D(w) = \frac{1}{2} H_{D,k}\left(\frac{\mathbf{1}}{N}\right) + H_{D_{K\times K}}(n),$$

where σ_k^2 is the variance of the equal weighted average returns in asset class k, and

$$H_{D_{K\times K}}(n) = \frac{1}{2}\sum_{k=1}^{K}\sum_{l=1}^{K} n_k\, n_l\, (\sigma_k^2 + \sigma_l^2 - 2\,\sigma_{kl}).$$

Since $H_{D,k}\left(\frac{\mathbf{1}}{N}\right)$ is constant, maximizing $H_D(w)$ is equivalent to maximizing $H_{D_{K\times K}}(n)$, which is portfolio RQE at the asset class level. Portfolio RQE is thus a useful tool to study diversification within and between asset classes.

Remark 4.3. *In (4.1), the first stage of decomposition is at the single-asset level and the second stage is at the level of assets class. Portfolio RQE can also be decomposed at N levels (see Rao, 2004). Therefore, in applications, portfolio RQE can be a useful tool to simultaneously study diversification at the country, industry, assets class level, etc.*

4.6.2 Time Decomposition.

Another possible decomposition of portfolio RQE is at the level of time periods. Consider the following example. Suppose that the matrix D is defined by (3.10). In that case, one can show that

$$H_D(w) = \sum_{i=1}^{N} w_i \|y_{it} - \overline{y}_t\|^2, \tag{4.2}$$

where $y_{it} = R_{it} - \mu_i$, $\overline{y}_t = \sum_{i=1}^{N} w_i y_{it}$ and $\|.\|$ is the Euclidean norm. By rearranging (4.2), one obtains

$$H_D(w) = \frac{\sum_{t=1}^{T} \sigma_t^2(w)}{T}, \tag{4.3}$$

where

$$\sigma_t^2(w) = \sum_{i=1}^{N} w_i (y_{it} - \overline{y}_t)^2,$$

is asset returns variance at period t. Using (4.3), one can measure period t's diversification contribution by

$$RQEC_t = \frac{\sigma_t^2(w)}{\sum_{t=1}^{T} \sigma_t^2(w)}.$$

$RQEC_t$ can be used to identify periods with higher diversification potential, as well as to compare two bear market periods and two bull market periods.

Remark 4.4. *$RQEC_t$ cannot be used to compare a bear market period with a bull market period. This is because dissimilarities between assets are low during extreme bear and bull market, and so is the value of $H_D(w)$. From the perspective of diversification, portfolio RQE therefore reveals that bull and bear markets are similar.*

4.7 Summary

Properties 1 to 5, as well as the decomposition properties in Sections 4.6.1 and 4.6.2 are deemed desirable for a measure of portfolio diversification. In particular, the Properties 1 to 5 are necessary for a measure to be portfolio diversification measure. This section has shown that portfolio RQE verifies all these requirements, and is a useful tool to study diversification while being consistent with the old adage "don't put all your eggs in one basket". Indeed, the old adage is the most used definition of portfolio diversification (see for example Markowitz et al., 2009, pp. 12) and is usually understood as meaning to spread eggs (dollars) across many baskets. However, this interpretation is misleading (Carli et al., 2014). To illustrate, consider one portfolio that allocates 100% of the wealth on asset A and a second portfolio of two assets A, B that allocates 50% of the wealth on each asset, with B a duplication of A. According to the traditional interpretation of the adage, the second portfolio appears more diversified than the first, but this is not true; spreading eggs across basket A and basket B is equivalent to putting all your eggs in basket A, since B is a duplication of A. This illusion is eliminated when proper account of asset dissimilarity is taken. In that context the old adage should rather be seen as the prescription to spread all your eggs (dollars) across many baskets while taking into account basket dissimilarity (as recommended implicitly in Markowitz (1952, p. 89)). More precisely, it should be seen as the prescription to spread your eggs (dollars) across many baskets by putting more eggs in more dissimilar baskets and fewer eggs in less dissimilar baskets. A portfolio diversification measure consistent with the old adage therefore needs to verify Properties 1 and 2, and Proposition 3.1. This is the case of portfolio RQE and this section strengthens our proposition that it is a valid measure of portfolio diversification.

5 Properties of RQE Portfolios

This section first establishes two equivalent definitions of RQEP, which we call Core Property 1 and Core Property 2. From these definitions, more intuitive interpretations of RQEP are provided and portfolio RQE diversification criteria are deduced. Second, this section establishes the conditions under which RQEP are mean-variance optimal, and shows how to use portfolio RQE when RQEP are mean-variance sub-optimal.

5.1 The Core Properties of RQEP

The first order conditions for optimization of portfolio RQE are

$$\sum_{j=1}^{N} d_{ij} w_j^{RQE} = \eta - \nu_i, \ \ \forall\, i = 1, ..., N, \tag{5.1}$$

$$w^\top \mathbf{1} = 1, \tag{5.2}$$

$$min(w_i, \nu_i) = 0, \ \ \forall\, i = 1, ..., N, \tag{5.3}$$

where η is the Lagrange multiplier of the constraint $w^\top \mathbf{1} = 1$, ν_i the Lagrange multiplier of the i^{th} non-negativity constraint and $\mathbf{1}$ is a column vector of ones. From the conditions (5.1)-(5.3), two equivalent definitions can be provided for RQEP. The first definition provides a very intuitive interpretation of the nature of RQEP and is established in Proposition 5.1. Similarly to Choueifaty et al. (2013), we call it the Core Property (1) of RQEP.

Proposition 5.1 (Core Property (1)). *RQEP is the portfolio such that:*

1. *the dissimilarity between any asset i that belongs to the RQEP and itself is equal to its portfolio RQE i.e.*

$$D_{H_D}\left(w^{RQE}, w^i\right) = H_D\left(w^{RQE}\right), \quad \forall\, i / w_i^{RQE} > 0. \tag{5.4}$$

2. *the dissimilarity between any asset j that does not belong to the RQEP and itself is smaller than its portfolio RQE i.e.*

$$D_{H_D}\left(w^{RQE}, w^j\right) < H_D\left(w^{RQE}\right), \quad \forall\, j / w_j^{RQE} = 0. \tag{5.5}$$

Proof of Proposition 5.1. See Appendix A.6. ■

Accordingly,

$$D_{H_D}\left(w^{RQE}, w^{j}\right) < D_{H_D}\left(w^{RQE}, w^{i}\right) \quad i \neq j, \tag{5.6}$$

and

$$D_{H_D}\left(w^{RQE}, w^{i}\right) = D_{H_D}\left(w^{RQE}, w^{k}\right), \quad \forall\, i, k \,/ w_i^{RQE},\, w_k^{RQE} > 0. \tag{5.7}$$

Equation (5.6) brings a justification to the concentration (in terms of the number of assets held) of RQEP. It shows that an asset can be effectively held even if it is not physically held. Portfolio RQE therefore can reduce transaction costs without lowering a portfolio's diversification. Equation (5.7) shows that, contrary to the equally-weighted portfolio and to the equally-risk contribution portfolio (in which the diversification is expressed in terms of weight and risk respectively) the diversification in RQEP is expressed in terms of dissimilarity between the single-asset portfolio and RQEP and can be defined as follows:

Definition 5.1 (Dissimilarity-Diversification Criteria). *A portfolio is RQE well-diversified if and only if:*

1. *it is as far as possible from all assets that belong to it.*
2. *it is "equidissimilar "from all assets that belong to it.*

The intuition behind this criteria is that in principle, a portfolio needs to be close to assets with desirable properties (such as high expected return and low risk in the mean-variance paradigm) and far from assets with undesirable ones (low expected return and high volatility). However, when there isn't enough information to distinguish assets with desirable properties from those with undesirable ones, the optimal attitude to adopt for a portfolio manager is to be neutral. Being neutral involves constructing a portfolio having the following properties: first, all assets that belong to the portfolio must be as far as possible from it. Second, all assets that belong to it must be "equidissimilar "from it. This attitude is natural and serves to protect RQEP from both asset individual shocks (second condition of Definition Definition 5.1) and mass destruction (first condition of Definition Definition 5.1). As a result, portfolio RQE diversification serves as protection against ignorance, but which kind of ignorance? We answer this question in the next sub-section using the mean-variance paradigm.

Equation (5.4) also shows that the more RQEP is diversified, the higher is the dissimilarity between RQEP and an asset held in it. This means that the philosophy behind RQE diversification is to beat the single-asset portfolios, the least diversified portfolios, which comes as no surprise considering Equation (3.4).

Core Property (1) is equivalent to the following alternative definition, which we call Core Property (2) of RQEP similarly to Choueifaty et al. (2013).

Proposition 5.2 (Core Property (2)). *RQEP is the portfolio such that the dissimilarity between any other portfolio and itself is smaller than or equal to the difference of their portfolio RQEs i.e.*

$$D_{H_D}\left(w^{RQE}, w\right) \leq H_D\left(w^{RQE}\right) - H_D(w).$$

Proof of Proposition 5.2. See Appendix A.7.■

Accordingly, the more diversified an arbitrary long-only portfolio w is, the smaller its dissimilarity from RQEP. In particular, when we have more than one RQEP, all have a dissimilarity of zero between themselves, so that all RQEP are equivalent. Holding one rather than any other does not affect the degree of diversification (or protection) offered by RQEP. Conventional wisdom however suggests that in this case the least concentrated (in terms of weights) must be preferred.

5.2 Optimality of RQEP in Mean-Variance Framework

This section discusses the choice of D in the mean-variance paradigm. We argue that the choice of D in MPT must depend on the quantity and the quality of available information as well as asset characteristics that the investor wants to take into account.

Within the mean-variance paradigm, the complete set of relevant asset characteristics are the vector of asset expected returns $\mu = (\mu_i)_{i=1}^N$, the vector of asset volatilities $\sigma = (\sigma_i)_{i=1}^N$ and the matrix of correlation between asset returns $\rho = (\rho_{ij})_{i,j=1}^N$, and the complete information set can be defined by $\mathcal{I} = \{\mu, \sigma, \rho\}$.

No Information Available ($\mathcal{I} = \emptyset$)

Assume that $\mathcal{I} = \emptyset$ i.e. no information is available. In that case, there is no reason to believe that, given any two pairs of assets (i, j) and (k, l), the dissimilarity between i

and j is greater or lesser than the dissimilarity between k and l. In such a situation, the dissimilarity matrix can be specified by (3.7) and since the Gini-Simpson index is equivalent to Markowitz's utility function when no information is available, (3.7) is mean-variance rational. In short, when no information is available the optimal choice of the dissimilarity matrix is (3.7) and portfolio RQE is mean-variance optimal.

Incomplete Information : case 1 $(\mathcal{I} = \{\rho\})$

Now assume that $\mathcal{I} = \{\rho\}$ i.e. only the available information is the correlation matrix (available without estimation errors). In that case, the dissimilarity matrix can be specified by (3.12) and one can show that portfolio RQE is equivalent to Markowitz's utility function. In short, when the only available information is ρ, the optimal dissimilarity matrix is (3.12) and portfolio RQE is now equivalent to portfolio variance and Markowitz's utility function. Moreover, one can also show that it is equivalent to the diversification ratio and portfolio normalized variance.

Incomplete Information : case 2 $(I = \{\sigma, \rho\})$

Now assume that $I = \{\sigma, \rho\}$ i.e. information about the covariance matrix is available (again without estimation errors). In that case, the dissimilarity matrix can be specified by (3.10). This specification is not systematically mean-variance rational, unless assets have the same volatility. The intuition behind this result is given by portfolio RQE diversification criteria (Definition 5.1). Given that σ represents an individual characteristic of assets, it is no longer optimal to be "equidissimilar "from assets held. Rather it is better to be close to assets with lower volatility and further way from assets with higher volatility. As a result, holding RQEP when $\mathcal{I} = \{\sigma, \rho\}$ is mean-variance sub-optimal and can be interpreted as an example of over diversification. Portfolio RQE must now be coupled with another measure to achieve better performance. Below, we present three examples to show how this can be achieved.

Example 1 (portfolio variance). *Consider portfolio variance*

$$\sigma^2(w) = w^\top \Sigma\, w.$$

where $\Sigma = (\sigma_{ij})_{i,j=1}^N$ *is the covariance matrix. Exploiting the definition of diversification* *return, we can rewrite $\sigma^2(w)$ as follows:*

$$\sigma^2(w) = w^\top \sigma^2 - H_D(w),$$

where D is specified by (3.10). Accordingly, the portfolio variance is a combination of portfolio RQE and the weighted average of assets variances. As a result, portfolio RQE can be combined with a function of asset variances under the information set $\mathcal{I} = \{\sigma, \rho\}$ to achieve better performance.

Example 2 (portfolio variance normalized). *Consider now portfolio variance normalized*

$$NV(w) = \frac{w^\top \Sigma\, w}{w^\top \sigma^2}$$

As we can observe, NV depends only on the elements of the covariance matrix Σ, so it is defined under the information set $S_I = \{\sigma, \rho\}$. It is straightforward to show that NV is equivalent to

$$\frac{1}{NV(w)} - 1 = \frac{w^\top \sigma^2 - w^\top \Sigma\, w}{w^\top \Sigma\, w}.$$

Accordingly, NV is a combination of portfolio RQE (with D specified by (3.10)) and portfolio variance and it can be interpreted as a risk-diversification trade-off where the risk is measured by portfolio variance and the diversification by portfolio RQE. NV can also be interpreted as risk-adjusted diversification return. This shows how portfolio RQE can be combined with portfolio variance under the information set $\mathcal{I} = \{\sigma, \rho\}$ to achieve better performance.

Example 3 (diversification ratio). *Finally, consider the diversification ratio*

$$DR(w) = \frac{w^\top \sigma}{\sqrt{w^\top \Sigma\, w}}.$$

As showing in Choueifaty et al. (2013), the optimal portfolios of DR (most diversified portfolio or MDP in short) can be obtained in two steps. The first step consists of the minimization of the objective function

$$C(w) = \frac{1}{w^\top \rho\, w},$$

and the second step consists to rescale the resultant weights by corresponding asset volatilities. Notice that the first step also consists of the maximization of portfolio RQE (with D

specified by (3.12)) : Therefore, the MDP is obtained by the combination of portfolio RQE and asset volatilities. As a result, under the information set $\mathcal{I} = \{\sigma, \rho\}$, a more efficient portfolio can be obtained by rescaling RQEP by asset volatilities.

Remark 5.1. *It is important to stress that when portfolio variance cannot be used as portfolio risk (see Markowitz, 2014, for more details), it is not systematically necessary to combine portfolio RQE with other portfolio performance measures to achieve a better performance. We plan to investigate this fact further in further research.*

Complete Information : $\mathcal{I} = \{\mu, \sigma, \rho\}$

Now assume that $\mathcal{I} = \{\mu, \sigma, \rho\}$ i.e. we have complete information. In that case, D can be specified as previously (Equation (3.10)). The specification remains non mean-variance rational however, unless assets have the same certain equivalent

$$\mathbb{E}(R_i) - c = \tau\, \sigma_i^2, \tag{5.8}$$

where c is a constant and τ is the coefficient of risk aversion. If we assume that a risk-free asset is available, with return R_{n+1}, it is straightforward to show that condition (5.8) remains valid with $c = R_{n+1}$:

$$\mathbb{E}(R_i) - R_{n+1} = \tau\, \sigma_i^2, \tag{5.9}$$

where τ is equal to the social risk aversion coefficient, and RQEP can be obtained through the two fund separation theorem.

However, even if the relation (5.9) can be justified theoretically by the results of Merton (1987) and Malkiel and Xu (2006)[8], it cannot hold due to the low volatility anomaly.[9]

[8] Malkiel and Xu (2006)'s model is a generalization of Merton (1987). The authors show that when some investors cannot hold the market portfolio, the remaining investors will also be unable to hold that portfolio. Therefore, idiosyncratic risk could be priced in part to compensate rational investors for an inability to hold the market portfolio.

[9] This anomaly says that portfolios of high volatility stocks underperformed those of low volatility stocks in term of risk-adjusted returns. For example, Baker et al. (2011) show that regardless of whether the risk is defined (beta or volatility) or whether all stocks or only large caps are considered, low-risk portfolios consistently outperformed high-risk portfolios over the period considered (January 1968-December 2008).

Moreover, one can show that (5.9) is not a non-arbitrage relation.[10] Accordingly, it is unlikely that portfolio RQE turns out to be mean-variance optimal when we have complete information. This is because it is no longer optimal to be "equidissimilar "from assets held. Rather, it is better to be close to assets with high expected return and low volatility. Therefore, under the information set $\mathcal{I} = \{\mu, \sigma, \rho\}$, portfolio RQE must be combined with another performance measure to achieve better performance. Below, we present such an example.

Example 4 (Markowitz's utility function). *Consider Markowitz's utility function (assuming that the risk free asset is not available)*

$$U(w) = w^\top \mu - \tau\, w^\top \Sigma\, w,$$

$U(w)$ can also be rewritten as

$$U(w) = \left(w^\top \mu - \tau w^\top \sigma^2\right) + \tau \left(w^\top \sigma^2 - w^\top \Sigma\, w\right) \tag{5.10}$$

where τ is the coefficient of the risk aversion. The first term of (5.10) represents the weighted average of asset certain-equivalents and the second term is portfolio RQE multiplied by the coefficient of the risk aversion (see particular case 3.3.2). Thus, Markowitz's utility function can be viewed as a combination of asset certain equivalents and portfolio RQE, a nice example of the use of portfolio RQE under complete information. As a result, portfolio RQE can be combined with portfolio certain-equivalent under the information set $\mathcal{I} = \{\mu, \sigma, \rho\}$ to achieve better performance.

Equation (5.10) also gives a new interpretation to the mean-variance model, providing answers to frequently asked questions in this paradigm: "Does a specific portfolio diversification measure exist in mean-variance model? "(see Fernholz, 2010) or "How can we quantify the lack of diversification of the Markowitz portfolios? "(see Pola, 2014b). From Equation (5.10), we can conclude, contrary to Fernholz (2010), that a specific measure of portfolio diversification does exist in the mean-variance model and this measure is portfolio

[10] Following Lee (2011), consider two assets i and j with identical risk-adjusted return $\frac{\mathbb{E}(r_i) - r_{N+1}}{\sigma_i^2}$. A new company, k, can be created by holding shares of i and j on the balance sheet such as the risk-adjusted return of k is higher than those of i and j unless the correlation between i and j is $+1$ and i and j have the same volatility, which means i and j are redundant. Therefore, (5.9) can be violated and an arbitrage opportunity exits unless all the assets of the universe is redundant.

RQE. The lack of diversification of Markowitz's portfolios can therefore be measured using portfolio RQE. This result reinforces our claim that portfolio RQE can be considered as an unifying approach for portfolio diversification.

Equation (5.10) also allows to interpret the coefficient of the risk aversion τ as the coefficient of the diversification aversion. Further, we can measure the diversification return in the mean-variance model by $\tau\left(w^\top \sigma^2 - w^\top \Sigma\, w\right)$, where the term in parenthesis is diversification benefit measured by portfolio RQE.

Imperfect Information

In general, the vectors μ, σ, ρ are set at their estimated values using the historical data, ignoring estimations risk. However, these risks are known to have a huge impact on Markowitz's portfolios. More precisely, they are the source of undesirable extreme weights and the poor out-of-sample performance of Markowitz portfolios (Chopra and Ziemba, 1993). In the RQE paradigm, the matrix D can be specified to take estimation risk into consideration.

To illustrate, consider the information set $\mathcal{I} = \{\rho\}$. As discussed above, the optimal choice of the dissimilarity matrix in this case is (3.12). Assume now that asset correlations are estimated with errors, so that (3.12) becomes sub-optimal. Denoting the specification (3.7) and (3.12) D_0 and D_1 respectively, we argue that the dissimilarity matrix can be specified as follows:

$$d_{\theta,ij} = (1-\theta)d_{1,ij} + \theta\, d_{0,ij}, \tag{5.11}$$

where parameter $\theta \in [0,1]$ takes into account estimation errors. When estimation errors are equal to zero, $\theta = 0$ and D_θ reduces to D_1, which is optimal. When estimation errors are very high, however $\theta = 1$ and D_θ reduces to D_0, which is optimal. This occurs because, under very high estimation errors the information set $\mathcal{I} = \{\rho\}$ reduces to $\mathcal{I} = \emptyset$.

In such case $d_{\theta,ij}$ can be interpreted as the expected dissimilarity, and the parameter θ represents the probability that the dissimilarity between assets i and j is equal to $d_{1,ij}$, while $1-\theta$ represents the probability that the dissimilarity between assets i and j is equal to $d_{0,ij}$. It can also be interpreted as the shrinking dissimilarity, whereby $d_{1,ij}$ is the guess, $d_{0,ij}$ is the shrinkage target, and θ is the shrinkage constant. The parameter θ can be calibrated following Ledoit and Wolf (2003). Below, we present an example where D_θ is used to take into account estimation errors.

Example 5 (Bouchaud's general free utility function). *Consider Bouchaud's general free utility*

$$F_q(w) = U(w) - \xi\,\frac{Y_q(w) - 1}{q - 1},$$

where U is Markowitz's utility function, $Y_q(w) = \sum_{i=1}^n w_i^q$ is a concentration measure, $-\frac{Y_q(w)-1}{q-1}$ is Patil and Taille (1982)'s diversity measure and q an integer greater than one. We consider the case $q = 2$. In that case, $-\frac{Y_q(w)-1}{q-1}$ coincides with the Gini-Simpson index, and Equation (5.10) implies that

$$F_2(w) = w^\top(\mu - \tau\sigma^2) + (\tau + \xi)\, w^\top D_\alpha\, w,$$

where $\theta = \frac{\tau}{\tau+\xi}$ and $D_\theta = \theta\, D_2 + (1-\theta)\, D_0$, with D_2 denotes a dissimilarity matrix specified by (3.10). Therfore, portfolio RQE is at the core of Bouchaud 's general free utility function where the diversification risk aversion coefficient is $\tau + \xi$, that is the sum of the standard risk aversion coefficient and the estimation risk aversion coefficient. As a result, portfolio RQE can be combined with portfolio certain equivalent under the imperfect information set $\mathcal{I} = \{\mu, \sigma, \rho\}$ to achieve better performance.

Remark 5.2. *The dissimilarity specification (5.11) can also be used when portfolio managers wants to handle future uncertainty.* [11] *In that case, $\theta = 0$ means that past performance is guarantee of future success, so there is no uncertainty going forward and the optimal dissimilarity matrix is the estimated D matrix. However, $\theta = 1$ means that past performance is no guarantee of future success, which means there is total uncertainty about the future. The optimal choice of the dissimilarity matrix in the latter case is D_0, defined above.*

5.3 Summary

In the light of these results, two conclusions obtain. First, portfolio RQE is a core around which many portfolio diversification measures and utility functions are built. This shows that portfolio RQE is a unifying approach to the measures of portfolio diversification. Sec-

[11] Even if μ, σ, ρ are estimated without error, since they are based on historical data, they represent past performances only. However, past performances are no guarantee of future success.

Table 1: List of Portfolios Considered

Portfolios	Abbreviation
RQE portfolio with $D = D_1$	$\mathrm{RQEP_{D_1}}$
RQE portfolio with $D = D_2$	$\mathrm{RQEP_{D_2}}$
RQE portfolio with $D = D_{0.5}$	$\mathrm{RQEP_{D_{0.5}}}$
RQE portfolio with $D = D_{0.8}$	$\mathrm{RQEP_{D_{0.8}}}$
Equally weight portfolio	EW
Equally risk contribution portfolio	ERCP
Most diversified portfolio	MDP
Capitalization weight index	CW
Market portfolio	Mkt

Notes. This table lists the various portfolios we compare. The MKT portfolio refer to the capitalization-weight portfolio of all New York Stock Exchange (NYSE), American Stock Exchange (AMEX), and National Association of Securities Dealers Automated Quotations (NASDAQ) firms (consult Kenneth R. French website : http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/Data_Library/f-f_factors.html for more details). The CW is the capitalization-weight index computed from industries annual size data available in the Excel file containing our reference universe data. D_1 and D_2 are dissimilarity matrix specified respectively by (3.12) and (3.10). $D_{0.5}$ and $D_{0.8}$ is the combination of D_2 and D_0 with *alpha* equal respectively to 0.5 and 0.8.

ond, portfolio RQE allows a more transparent interpretation of many diversification measures and utility functions. For example, the Gini-Simpson index diversification protects from ignorance about assets future returns, assets expected returns and assets covariance matrix. Portfolio RQE (Equation (3.7)) protects from ignorance about assets future returns, assets expected returns and assets volatility. Diversification ratio and portfolio normalized variance protects from ignorance about assets future returns and assets expected returns. Markowitz's utility function protects from ignorance about assets future returns. Bouchaud's utility function protects from ignorance about assets future returns and estimation risk. Diversification return or excess growth rate can be view as a protection from ignorance about assets certain equivalent.

6 Portfolio RQE and Mass Destruction Protection : Illustration

This section provides an empirical illustration of the capacity of RQEP to protect from mass destruction as discussed in Section 3 and 5. We compare, both in-sample and out-of-sample, four RQEP with the most popular existing diversified portfolios using two measures of portfolio performance: its return during bear market periods and its market beta. To conduct the in-sample comparisons, we use the Fama-French forty-nine industry portfolios dataset of equal weighted annual asset average returns with forty-four years observed period from 1970 to 2013 as reference universe. For the out-of-sample comparisons, we use the Fama-French forty-nine industry portfolios data set of equal weighted daily asset average

Table 2: Portfolio Expected Average returns Comparison During Bear Market Periods (BMP) In-Sample

BMP	RQEP				MDP	ERCP	EWP	CW	MKT	MVP
	D_1	D_2	$D_{0.5}$	$D_{0.8}$						
1973	0.030	0.144	0.041	-0.138	-0.058	-0.296	-0.325	-0.300	-0.193	-0.161
1974	-0.054	-0.066	-0.131	-0.182	-0.071	-0.194	-0.216	-0.230	-0.277	-0.127
1977	0.166	0.149	0.175	0.206	0.167	0.215	0.228	0.223	-0.031	0.127
1981	-0.028	-0.179	-0.129	-0.060	0.039	0.030	0.019	-0.024	-0.034	0.121
1987	0.115	0.157	0.128	0.029	0.069	-0.061	-0.070	-0.061	0.016	-0.037
1990	-0.173	-0.210	-0.190	-0.193	-0.149	-0.195	-0.207	-0.182	-0.061	-0.062
2000	0.440	0.239	0.191	0.105	0.478	0.081	0.014	-0.082	-0.115	0.423
2001	0.296	0.282	0.199	0.236	0.302	0.245	0.243	0.208	-0.113	0.156
2002	0.280	0.391	0.212	0.079	0.161	-0.042	-0.056	-0.134	-0.211	-0.064
2008	-0.345	-0.439	-0.456	-0.459	-0.296	-0.420	-0.452	-0.438	-0.367	-0.210
2011	-0.078	-0.160	-0.153	-0.127	-0.033	-0.083	-0.102	-0.070	0.005	0.118
Average	0.059	0.027	-0.010	-0.045	0.055	-0.065	-0.084	-0.099	-0.125	0.025

Notes. This table compares the expected average returns of considered portfolios in-sample during bear market periods. We consider as reference universe the Fama-French forty-nine industry portfolios data set of equal weighted annual asset average returns with forty-four years observed periods from 1970 to 2013. This reference universe gives a complete representation of the U.S. stock market.

returns with observed period from 07-01-1969 to 12-31-2013 as the reference universe.

We identify the bear market periods as period during which the Standard & Poor's index (S & P 500) fell at least 20%. We estimate the covariance matrix for RQEP, MVP, MDP and ERC using Ledoit and Wolf (2003)'s shrinkage estimator, where the shrinkage target is obtained from a one-factor model and the factor is equal to the cross-sectional average of all the random variables (assets return). [12]

6.1 In-Sample Comparison

We begin by computing portfolios returns. Given a portfolio w (with the exception of the MKT), we compute the in-sample returns vector as $R(w) = R\,w^\top$, where R is $T \times N$ assets (industries) returns matrix, with $T = 43$ the number of periods and $N = 49$ the number of industries. The in-sample return of a portfolio w at the bear market periods t is then given by $R_t(w) = R_t\,w^\top$. Specially, in the case of the CW, the in-sample returns vector is given by $R_t(w) = R_t\,w_t^\top$, where w_t is the CW index at period t. MKT returns are taken from Kenneth R. French's website. The in-sample portfolio returns in all identified bear market periods are reported in Table 2. Since a portfolio w is not rebalanced, we only interpret the average of portfolio return during all identified bear market periods. As we can see,

[12]We do this using the code available at https://r-forge.r-project.org/scm/viewvc.php/pkg/ExpectedReturns/man/?root=expectedreturns&pathrev=2.

Table 3: Fama-French Yearly Regression Coefficient, 1970-2013 In-Sample

Portfolios	MKT	SMB	HML	ALPHA	R^2
$RQEP_{D_1}$	0.603***	0.750***	0.322	0.057*	46.91%
$RQEP_{D_2}$	0.604***	1.232***	-0.180	0.074	44.3%
$RQEP_{D_{0.5}}$	0.760***	1.177***	-0.112	0.062***	57.63%
$RQEP_{D_{0.8}}$	0.891***	1.169***	0.052	0.035	77.30%
EWP	1.016***	1.134***	0.224**	0.004	90.40%
CW	1.065***	1.159***	-0.033	0.028**	92.70%
MDP	0.657***	0.630***	0.422***	0.050**	57.94%
ERCP	0.963***	1.018***	0.286***	0.006	90.06%
MVP	0.703***	0.229**	0.532***	0.019	76.33%

Notes. This table reports the results of Fama-French 3-factor model regressions for each considered portfolio in-sample. The principal goal is to rank the portfolios according to their market risk (coefficient of the factor MKT). The factor SMB (Small Minus Big) measures portfolio size premium and the factor HML (High Minus Low) measures portfolio value premium.
Significance levels: $***$ = 1%; $**$ = 5%; $*$ = 10%.

$RQEP_{D_1}$, $RQEP_{D_2}$ and the MDP provide the best protection. The result of the MDP is without surprise, since the MDP holds the same assets with $RQEP_{D_1}$ (the two portfolios are exposed to the same risk factors). Therefore, we can conclude that RQEP is a cautious portfolio offering better protection from mass destruction.

Next, we obtain the market beta of a portfolio w by running Fama-French 3- factors regressions using its expected returns vector, $R(w)$. The objective is to rank portfolios according to their market beta. The results of these regressions are reported in Table 3. As we can see, $RQEP_{D_1}$ and $RQEP_{D_2}$ have the lowest market exposure, followed by the MDP and the MVP. Even if $RQEP_{D_{0.5}}$ and $RQEP_{D_{0.8}}$ are more beta-risky than the MDP and the MVP, they still have less beta-risk than the EWP, the CW and the MKT. As a result, RQEP is in general less beta-risky and is a cautious portfolio offering better protection from mass destruction.

We can also see that all RQEP have positive SMB factor exposure, so they are less biased toward large capitalizations stocks. Unsurprisingly, the MVP, the MDP and the ERCP have positive and significant HML factor exposure, in opposite to RQEP. This shows that RQEPs are not low volatility strategies. RQEP except $RQEP_{D_{0.8}}$ and the MDP are the portfolios which exhibit lowest R^2, showing thus that much of their performances are not explained by the market capitalization index and the other two factors. This result also suggests that RQEP is more risk-factor diversified than the other portfolios considered. This comes as no surprise, since it maximizes the effective number of independent risk factors.

Table 4: Portfolio Expected Average returns Comparison During Bear Market Periods (BMP) Out-Of-Sample

BMP	RQEP				MDP	ERCP	EWP	CW	MKT	MVP
	D_1	D_2	$D_{0.5}$	$D_{0.8}$						
1973	-0.090	-0.276	-0.261	-0.313	-0.074	-0.297	-0.323	-0.298	-0.193	-0.154
1974	-0.089	-0.197	-0.167	-0.179	-0.071	-0.160	-0.172	-0.194	-0.277	-0.158
1977	0.273	0.281	0.288	0.271	0.273	0.253	0.256	0.237	-0.031	0.165
1981	-0.095	-0.243	-0.165	-0.080	0.043	0.058	0.021	-0.032	-0.034	0.170
1987	0.186	0.381	0.232	0.097	0.140	0.026	0.014	0.019	0.016	0.052
1990	-0.103	-0.174	-0.120	-0.091	-0.092	-0.079	-0.076	-0.068	-0.061	-0.025
2000	0.304	0.636	0.351	0.212	0.258	0.150	0.139	-0.011	-0.115	0.180
2001	0.478	0.763	0.690	0.500	0.374	0.387	0.413	0.366	-0.113	0.323
2002	0.239	0.391	0.276	0.158	0.145	0.087	0.045	-0.038	-0.211	0.208
2008	-0.228	-0.261	-0.289	-0.344	-0.224	-0.361	-0.375	-0.353	-0.367	-0.287
2011	-0.029	-0.178	-0.134	-0.116	-0.011	-0.067	-0.083	-0.041	0.005	0.002
Average	0.076	0.101	0.063	0.010	0.069	-0.0002	-0.012	-0.037	-0.125	0.043

Notes. This table compares the expected average returns of considered portfolios out-of-sample during bear market periods. We consider as reference universe the Fama-French forty-nine industry portfolios data set of equal weighted daily asset average returns with observed periods from 01-07-1969 to 31-12-2013. This reference universe gives a complete representation of the U.S. stock market.

6.2 Out-of-Sample Comparison

We use a one year rolling-window of daily returns to generate out-of-sample portfolio returns. We compute portfolios annual returns from daily returns. The results are reported in Table 4. As we can observe, $\mathrm{RQEP_{D_1}}$ and $\mathrm{RQEP_{D_2}}$ give best protection during the bear market periods 1973, 1977, 1987, 2000, 2001 and 2002. The poor protection offered by $\mathrm{RQEP_{D_{0.5}}}$ and $\mathrm{RQEP_{D_{0.8}}}$ comes from the fact that the matrix D_α, $\alpha = 0.5, 0.8$ is the average of the matrix D_2 and D_0 and its performance is affected by that of EWP. The MVP offers the best protection during the bear market periods 1981 and 1990, while the MDP dominates during the bear market periods 1973, 1974 and 2008. The MKT offers the best protection during the bear market periods 2011. On average however $\mathrm{RQEP_{D_1}}$ and $\mathrm{RQEP_{D_2}}$ still offer the better protection.

Next, we obtain the market risk of a portfolio w by running Fama-French 3 factors model regressions using its expected returns vector, $R(w)$. The results of the regression are reported in Table 5. We find same results as previously.

Table 5: Fama-French Yearly Regression Coefficient, 1970-2013 Out-Of-Sample

Portfolios	MKT	SMB	HML	ALPHA	R^2
$RQEP_{D_1}$	0.763***	1.130***	0.165	0.123***	55.74%
$RQEP_{D_2}$	0.656**	1.387***	-0.070	0.121**	39.59%
$RQEP_{D_{0.5}}$	0.777***	1.307***	0.007	0.113***	53.07%
$RQEP_{D_{0.8}}$	0.972***	1.183***	0.133	0.094***	70.48%
EWP	1.102***	1.094***	0.277*	0.083***	79.48%
CW	1.140***	1.082***	0.006	0.095***	82.76%
MDP	0.795***	1.000***	0.306	0.115***	63.46%
ERCP	1.075***	1.038***	0.353**	0.084***	79.85%
MVP	0.865***	0.354***	0.610***	0.050***	76.64%

Notes. This table reports the results of Fama-French 3-factor model regressions for each considered portfolio out-of-sample. We find same results as previously. Significance levels: $***$ = 1%; $**$ = 5%; $*$ = 10%.

In short, RQEP offers a strategy which better protects from mass destruction than the most quoted diversified portfolios, in particular when the dissimilarity matrix is specified by D_1 or D_2. This result is valid both in-sample and out-of-sample, and suggests that RQEP is more risk-factor diversified than the most quoted diversified portfolios.

7 Conclusion

This paper has introduced a new portfolio diversification measure, portfolio Rao (1982b)'s Quadratic Entropy (RQE). We have first shown that portfolio RQE verifies many properties deemed desirable for a measure of portfolio diversification. Second, we have shown that portfolio RQE is a unifying approach that represents the core around which many existing portfolio diversification measures and utility functions are built, among them the diversification ratio (DR), the portfolio variance normalized (NV), the diversification return (Dr) or the excess growth rate, the Gini-Simpson index (GS), the return gap (RG), Markowitz's utility function and Bouchaud's general free utility. Third, we have shown that portfolio RQE is a flexible but formal approach for fund managers to develop new diversified portfolios taking into account various asset characteristics. Fourth, we have provided theoretical evidence that holding a RQE optimal portfolio (RQEP) provides protection from mass destruction and shown that this protection is substantially by comparing four RQEP with the most popular existing diversified portfolios, both in-sample and out-of-sample.

The paper has also established the conditions under which RQEP is mean-variance optimal. We have proved that RQEP is mean-variance optimal in two cases. The first case is when no information is available, and portfolio RQE is equivalent to the Gini-Simpson index, accordingly to Markowitz's utility function. The second case is when the only available information is the correlation matrix. In that case, portfolio RQE is equivalent to portfolio variance, accordingly to Markowitz's utility function. Therefore, in the mean-variance paradigm, holding RQEP protects against ignorance about assets individual characteristics which are asset expected returns and volatility, and asset future returns.

When information on assets individual characteristics are known, RQEP is no longer mean-variance optimal. It is better to be close to assets with desirable individual characteristics, and holding RQEP, in that case creates over-diversification. A better performance can be achieved combining portfolio RQE and other portfolio performance measures. We have shown that portfolio variance, normalized portfolio variance, diversification ratio, Markowitz's utility function and Bouchaud's general free utility function are an example of the use of portfolio RQE in that case. As a result, the interpretation of the above diversification measures and utility functions become more transparent with portfolio RQE.

In extending Rao's Quadratic Entropy to portfolio selection, our objective was to develop a framework to measure unambiguously the extent of a portfolio's diversification. We argue that a portfolio RQE should be considered a strong candidate for being an "ideal "portfolio diversification measure. The adoption of RQEP by investors, and their trustees should help them navigate the uncertain world of portfolio selection.

This is a first study. Further research will include : (i) a deeper investigation of the choice of the dissimilarity matrix D; (ii) a deeper comparison of RQE portfolios with the most popular existing diversified portfolios and (iii) the comparison of mean-RQE and mean-variance models.

A Proofs of Propositions

A.1 Proof of Proposition 3.1

Consider the first order conditions of problem (3.6) (Propositions 4.1 and 4.3). Since we consider only assets held by RQEP, 4.1 becomes

$$\sum_{j=1}^{N_{w^{RQE}}} d_{ij} w_j^{RQE} = \eta, \quad \forall\, i = 1, ..., N_{w^{RQE}}. \tag{A-1}$$

Summing (A-1) over i gives

$$\sum_{j=1}^{N_{w^{RQE}}} w_j^{RQE} DC_j \left(w^{RQE}\right) = \eta\, N_{w^{RQE}}, \quad \forall\, i = 1, ..., N_{w^{RQE}}. \tag{A-2}$$

Equation (A-2) completes the proof of both the first and the second points, since $\eta = 2\, H_D \left(w^{RQE}\right)$.

A.2 Proof of Proposition 4.1

Let w be a portfolio and w^i be the i^{th} single asset portfolio ($w_i^i = 1$ and $w_j^i = 0,\ i \neq j$). First, since $w_i \geq 0,\ \forall i = 1, ..., N$ and $d_{ij} \geq 0,\ \forall i, j = 1, ..., N,\ H_D(w) \geq 0$. Second, $H_D(w^i) = d_{ii} = 0$. Therefore,

$$H_D(w) \geq H_D(w^i), \quad \forall i = 1, ..., N.$$

A.3 Proof of Proposition 4.2

Consider a universe of assets $U = \{A_i\}_{i=1}^N$ such as $A_i = A,\ \forall\, i = 1, ..., N$ and w a portfolio. Since $A_i = A,\ \forall\, i = 1, ..., N$, assets are perfectly similar. This implies that $d_{ij} = 0,\ \forall\, i, j = 1, ..., N$. Then

$$H_D(w) = 0.$$

A.4 Proof of Proposition 4.3

Let $U = \{A_i\}_{i=1}^N$ a universe of N assets. Consider a new universe $\boldsymbol{U} = U \cup \{B\}$ such as the asset B is duplicated i.e. there is an asset $k \in \{1, ..., N\}$ such as $B = A_k$. We denote by w^{RQE} and $\boldsymbol{w^{RQE}}$ the weights of RQEP associated to U and $\boldsymbol{U}$ respectively, and by H_D and $H_{\boldsymbol{D}}$ portfolio RQE associated to U and $\boldsymbol{U}$, respectively. H_D and $H_{\boldsymbol{D}}$ are defined as follows

$$H_D(w) = \sum_{i,j=1}^{N} d(A_i, A_j) w_{A_i} w_{A_j}, \tag{A-3}$$

and

$$H_{\boldsymbol{D}}(w) = \sum_{i,j\neq k}^{N} \boldsymbol{d}(A_i, A_j) w_{A_i} w_{A_j} + \sum_{i=1}^{N} \boldsymbol{d}(A_k, A_i) w_{A_k} w_{A_i} + \sum_{i=1}^{N} \boldsymbol{d}(B, A_i) w_B w_{A_i}. \tag{A-4}$$

w^{RQE} and $\boldsymbol{w^{RQE}}$ are defined as follows

$$w^{RQE} \in \underset{w \in W}{\arg\operatorname{Max}}\, H_D(w), \tag{A-5}$$

and

$$\boldsymbol{w^{RQE}} \in \underset{w \in \mathbf{W}}{\arg\operatorname{Max}}\, H_{\boldsymbol{D}}(w). \tag{A-6}$$

Since $B = A_k$, $\boldsymbol{d}(A_k, A_i) = \boldsymbol{d}(B, A_i), \ \forall\, i = 1, ..., N$. Then, (A-6) can be rewritten as follows

$$\boldsymbol{w^{RQE}} \in \underset{w \in \mathbf{W}}{\arg\operatorname{Max}} \sum_{i,j\neq k}^{n} \boldsymbol{d}(A_i, A_j) w_{A_i} w_{A_j} + \sum_{i=1}^{N} \boldsymbol{d}(A_k, A_i) w_{A_i} (w_{A_k} + w_B). \tag{A-7}$$

Let $w_1^* = (w_{A_1}^{RQE}, ..., w_{A_{k-1}}^{RQE}, w_{1,A_k}^*, w_{A_{k+1}}^{RQE}, ..., w_{A_N}^{RQE}, w_{1,B}^*)$ and $w_2^* = (\boldsymbol{w_{A_1}^{RQE}}, ..., \boldsymbol{w_{A_{k-1}}^{RQE}}, \boldsymbol{w_{A_k}^{RQE}} + \boldsymbol{w_{B}^{RQE}}, \boldsymbol{w_{A_{k+1}}^{RQE}}, ..., \boldsymbol{w_{A_N}^{RQE}})$ such as $w_{1,A_k}^* + w_{1,B}^* = w_{A_k}^{RQE}$. It is straightforward to show that

$$H_D\left(w^{RQE}\right) = H_{\boldsymbol{D}}(w_1^*) = H_D(w_2^*) = H_{\boldsymbol{D}}\left(\boldsymbol{w^{RQE}}\right). \tag{A-8}$$

Equation (A-8) shows that portfolio RQE and RQEP are both duplicate invariant.

A.5 Proof of Proposition 4.4

Let U a universe of N assets and $\boldsymbol{U}$ a universe of $N+1$ assets, derived from U increasing the size of U from N to $N+1$. Denote w^{RQE} RQEP weights of U and $\boldsymbol{w^{RQE}}$ that of $\boldsymbol{U}$.

Consider a portfolio $w = \left(w^{RQE}, 0\right)$. Portfolio w is an element of $\mathbf{W}$ of $\boldsymbol{U}$, so

$$H_{\boldsymbol{D}}\left(\boldsymbol{w^{RQE}}\right) \geq H_{\boldsymbol{D}}\left(w\right).$$

Since $H_{\boldsymbol{D}}\left(w\right) = H_D\left(w^{RQE}\right)$,

$$H_{\boldsymbol{D}}\left(\boldsymbol{w^{RQE}}\right) \geq H_D\left(w^{RQE}\right).$$

A.6 Proof of Proposition 5.2

First, we prove that RQEP respects the Core Property (2). Consider a compact form of (5.1)

$$D\, w^{RQE} = \eta \mathbf{1} - \nu. \tag{A-9}$$

Multiplying (A-9) by $\left(w^{RQE}\right)^{\top}$ gives

$$\eta = \left(w^{RQE}\right)^{\top} D\, w^{RQE} = 2\, H_D\left(w^{RQE}\right). \tag{A-10}$$

Substituting η in (A-9) and multiplying on the left the resulting equation by $w^{\top}$ lead to

$$w^{\top} D\, w^{RQE} = 2\, H_D\left(w^{RQE}\right) - w^{\top}\nu. \tag{A-11}$$

Subtracting $H_D\left(w^{RQE}\right)$ and $H_D(w)$ both from the left and right term of (A-11) lead to

$$2\, H_D\left(w^{RQE}, w\right) - H_D\left(w^{RQE}\right) - H_D(w) = H_D\left(w^{RQE}\right) - H_D(w) - w^{\top}\nu. \tag{A-12}$$

The result follows since $\nu \geq 0$.

We now prove that a portfolio w^* that respects the Core Property (2) necessarily maximizes portfolio RQE. Assume that w^* respects Core Property (2). In that case,

$$D_{H_D}\left(w^*, w\right) \leq H_D\left(w^*\right) - H_D(w). \tag{A-13}$$

Since $D_{H_D}\left(w^*, w\right)$ is a dissimilarity, we have $D_{H_D}\left(w^*, w\right) \geq 0$. Then $H_D\left(w^*\right) - H_D(w) \geq 0$. The results follows.

A.7 Proof of Proposition 5.1

First, we prove that RQEP respects the Core Property (1) i.e. Equations (5.4) and (5.5) of Proposition 5.1. Recall that

$$D_{H_D}\left(w^{RQE}, w\right) = H_D\left(w^{RQE}\right) - H(w) - w^\top \nu. \tag{A-14}$$

Assume that w is a i^{th} single asset portfolio. Then (A-14) becomes

$$D_{H_D}\left(w^{RQE}, w^i\right) = H_D\left(w^{RQE}\right) - \nu_i \tag{A-15}$$

where w^i is a i^{th} single asset portfolio i.e. $w_i^i = 1$. If asset i is held in RQEP, (A-15) coincides with (5.4) of Proposition 5.1. Otherwise, (A-15) coincides with (5.5) of Proposition 5.1. Therefore RQEP respects the Core Property (1).

Now we prove that any portfolio that respects the Core Property (1) necessary maximizes portfolio RQE. Assume that w^* respect the Core Property (1). Then,

$$D_{H_D}\left(w^*, w^i\right) \leq H_D\left(w^*\right), \quad \forall\ i = 1, ..., N. \tag{A-16}$$

Equation (A-16) can be rewritten as

$$2\sum_{j=1}^{N} d_{ij} w_j^* - H_D(w^*) \leq H_D\left(w^*\right). \tag{A-17}$$

Multiplying (A-17) by w_i and summing over i gives

$$2H_D(w, w^*) - H_D(w^*) \leq H_D\left(w^*\right). \tag{A-18}$$

By subtracting $H_D(w)$ both to right hand side and left hand side, we have

$$D_{H_D}\left(w, w^*\right) \leq H_D(w^*) - H_D(w). \tag{A-19}$$

Then w^* respect the Core Property (2). Therefore, w^* maximizes portfolio RQE.

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